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A collocation approach for solving fractional pantograph differential equations

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ABSTRACT

In this paper, a new numerical method for solving fractional pantograph differential equations is presented. The transformation matrix of Bessel polynomials to Taylor polynomials and Taylor operational matrix of fractional integration are used to transform the equation to a system of algebraic equations. Illustrative examples are included to demonstrate the validity and applicability of the technique.

KEYWORDS: Bessel polynomials, Fractional pantograph differential equations, Operational matrix of fractional order.

1 INTRODUCTION

In recent years use of fractional-order derivative going very strongly in engineering and life sciences and also in other area of sciences such as thermal systems, turbulence, image processing, fluid flow, mechanics, viscoelastic and other areas of applications (Bai et al. 2007), (Miller et al., 1993), (Podlubny, 2002). Recently, the paper (Yzbasi, 2013) presented a collocation method based on the Bernstein polynomials for the fractional Riccati type differential equations, the authors (Kazem et al., 2013), introduced fractional-order Legendre functions for solving fractional-order differential equations, the authors (Rahimkhani et al., 2017), applied generalized fractional-order Bernoulli wavelet for solving fractional pantograph differential equations. In this paper, we consider the fractional pantograph differential equation.

$$\begin{cases} D^{\upsilon}u(t) = F(D^{\gamma_0}u(q_0t), D^{\gamma_1}u(q_1t), \cdots, D^{\gamma_l}u(q_lt)), \qquad m-1<\gamma\le m, \\ \qquad\qquad u^{(i)}(0)=\lambda_i, \qquad i=0,1,\cdots,m-1, \end{cases} \tag{1}$$

here, $0<q_j\le 1,$ and $0\le\gamma_j<\gamma\le m,\ j=0,1,\cdots,l.$

2 PRELIMINARIES AND NOTATIONS

We give some basic definitions and properties of the fractional calculus theory, which are used further in this paper (Podlubny, 2002).

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0,$ of a function f is defined as

$$I^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y(t)dt, \quad x \geq 0,$$
$$I^0 f(x) = f(x).$$

Definition 2.2. The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$_*D^{\alpha} f(x) = I^{m-\alpha} {}_*D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-1} f^{(m)}(t)dt,$$

for $m-1 \leq \alpha < m, m \in \mathrm{N}, x > 0,$ where $D = \dfrac{d}{dt}.$

3 DESCRIPTION OF THE METHOD

The m-th degree truncated Bessel polynomials of first kind are defined by (Yuzbasi et al., 2012)

$$J_n(t) = \sum_{k=0}^{\left[\frac{N-n}{2}\right]} \frac{(-1)^k}{k!(k+n)!} (\frac{t}{2})^{2k+n}, \quad 0 \leq t < \infty, \tag{2}$$

where N is chosen the positive integer so that $N \geq n$ and $n = 0,1,\cdots,N.$ We can transform the Bessel polynomials of first kind to in N-th degree Taylor basis functions. In matrix form as

$$J(t) \simeq DT(t), \tag{3}$$

D is the transformation matrix, which defined in (Yuzbasi et al., 2012) and

$$J(t) = \left[J_0(t), J_1(t), \cdots, J_N(t)\right]^T, \qquad T(t) = \left[T_0(t), T_1(t), \cdots, T_N(t)\right]^T.$$

To solve Eq. (1) with conditions, we assume the highest order of derivative is 2,

$$u''(t) \simeq A^T J(t) = A^T DT(t), \tag{4}$$

where

$$A = \left[a_0, a_1, \cdots, a_N\right]^T.$$

By integrating Eq. (4) of order 2, we get

$$u'(t) \simeq A^T D \int_0^t T(\eta) d\eta + u'(0) = tA^T DLT(t) + \lambda_1, \tag{5}$$

and

$$u(t) \simeq A^T DL \int_0^t \eta T(\eta) d\eta + tu'(0) + u(0) = t^2 A^T DLST(t) + t\lambda_1 + \lambda_0, \tag{6}$$

where

$$L = diag(1, \frac{1}{2}, \cdots, \frac{1}{N+1}), \quad S = diag(\frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{N+2}).$$

For $0 < \gamma \leq 1,$ integrating from Eq. (5), we get

$$D^{\gamma} u(t) \simeq I^{1-\gamma}(u'(t)) = t^{1-\gamma} A^T DL \xi_N^{\gamma} T(t) + \frac{\Gamma(1)}{\Gamma(2-\gamma)} \lambda_1 t^{1-\gamma}. \tag{7}$$

Also, by using Eq. (4) for $1 < \gamma \leq 2,$ we obtain

$$D^{\gamma} u(t) \simeq I^{2-\gamma}(u''(t)) = t^2 A^T DLS \tilde{\xi}_N^{\gamma} T(t), \tag{8}$$

where

$$\xi_N^{\gamma} = diag(\frac{\Gamma(1)}{\Gamma(2-\gamma)}, \frac{\Gamma(2)}{\Gamma(3-\gamma)}, \cdots, \frac{\Gamma(N+1)}{\Gamma(N+2-\gamma)}),$$

$$\tilde{\xi}_N^{\gamma} = diag(\frac{\Gamma(2)}{\Gamma(3-\gamma)}, \frac{\Gamma(3)}{\Gamma(4-\gamma)}, \cdots, \frac{\Gamma(N+2)}{\Gamma(N+3-\gamma)}).$$

Here, by substituting Eqs. (4)-(8) in Eq. (1), we obtain a system of algebraic equations. Then, we collocate this system at the following points

$$t_i = \frac{2i-1}{2(N+1)}, \qquad i = 1,2,\cdots, N+1,$$

which can be solved this system for the unknown vector A by using Newton's iterative method.

4 ERROR ESTIMATION

In this section, we investigate the convergence analysis of our proposed method. We assume that $f(t)$ is a sufficiently smooth function on $[0,1]$ and $p_N(x)$ is the interpolating polynomial to f at points t_i, where t_i, $i = 0,1,\cdots, N$ are the roots of the (N + 1)-degree shifted Chebyshev polynomial in $[0,1]$, then we have (Podlubny, 2002)

$$f(t) - p_N(t) = \frac{f^{(N+1)}(\xi)}{(N+1)!} \prod_{n=0}^{N} (t - t_n), \quad \xi \in [0,1], \tag{9}$$

therefore, we obtain

$$| f(t) - p_N(t) | \leq \frac{M_N}{2^{2N+1}(N+1)!}, \tag{10}$$

where $M_N = \max_{t\in[0,1]} | f^{(N+1)}(t) |$.

Theorem 4.1. Suppose $u(t) \in C^{N+1}[0,1]$ and $u_N(t) = A^T J(t)$ be the approximate solution obtained by the present method in previous section. If $\tilde{u}_N(t) = \tilde{A}^T \tilde{J}(t)$ be the Bessel polynomials of first kind expansion of the exact solution $u(t)$, where

$$\tilde{J}(t) = \left[\tilde{J}_0(t), \tilde{J}_1(t), \cdots, \tilde{J}_N(t)\right]^T, \qquad \tilde{A} = \left[\tilde{a}_0, \tilde{a}_1, \cdots, \tilde{a}_N\right]^T,$$

and

$$\tilde{J}_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n)!} (\frac{t}{2})^{2k+n}, \quad 0 \leq t < \infty,$$

where the set of Bessel polynomials of first kind $\tilde{J}_n(t)$ in $L^2[0,1]$ is orthogonal with respect to the weight function $w(t) = t$,

$$\int_0^1 w(t)[\tilde{J}_n(t)]^2 \, dt = \frac{1}{2}[\tilde{J}_{n+1}(1)]^2.$$

Then obtain the upper bound of the error, as

$$\| f(t) - u_N(t) \|_{L_w^2} \leq \frac{M_N}{2^{2N+\frac{3}{2}}(N+1)!} + \| \tilde{A} - A \|_2 \, \theta_N + \| A \|_2 \, \omega_N, \tag{11}$$

where

$$\theta_N = \left[\sum_{n=0}^{N} \frac{1}{2}[\tilde{J}_{n+1}(1)]^2\right]^{\frac{1}{2}}, \qquad \omega_N = \left[\sum_{n=0}^{N} \sum_{k=[\frac{N-n}{2}]}^{\infty} \frac{1}{(k!(k+n)!2^{2k+n})^2(4k+2n+2)}\right]^{\frac{1}{2}}.$$

Proof: We can write

$$\| f(t) - u_N(t) \|_{L_w^2} \leq \| f(t) - \tilde{u}_N(t) \|_{L_w^2} + \| \tilde{u}_N(t) - u_N(t) \|_{L_w^2}, \tag{12}$$

since $\tilde{u}_N(t)$ is the best approximation of $u(t)$, we have

$$\begin{aligned} \| f(t) - \tilde{u}_N(t) \|_{L_w^2} &= \left(\int_0^1 | f(t) - \tilde{A}^T \tilde{J}(t) |^2 \, tdt \right)^{\frac{1}{2}} \leq \left(\int_0^1 | f(t) - p_N(t) |^2 \, tdt \right)^{\frac{1}{2}} \\ &\leq \left(\frac{M_N^2}{(2^{2N+1}(N+1)!)^2} \int_0^1 tdt \right)^{\frac{1}{2}} \leq \frac{M_N}{2^{2N+\frac{3}{2}}(N+1)!}. \end{aligned} \tag{13}$$

Also, we have

$$\begin{aligned} \| \tilde{u}_N(t) - u_N(t) \|_{L_w^2} &= \| \sum_{n=0}^{N} \tilde{a}_n \tilde{J}_n(t) - \sum_{n=0}^{N} a_n J_n(t) \|_{L_w^2} \\ &\leq \| \sum_{n=0}^{N} \tilde{a}_n \tilde{J}_n(t) - \sum_{n=0}^{N} a_n \tilde{J}_n(t) \|_{L_w^2} + \| \sum_{n=0}^{N} a_n \tilde{J}_n(t) - \sum_{n=0}^{N} a_n J_n(t) \|_{L_w^2} \\ &\leq \| \sum_{n=0}^{N} (\tilde{a}_n - a_n) \tilde{J}_n(t) \|_{L_w^2} + \| \sum_{n=0}^{N} a_n (\tilde{J}_n(t) - J_n(t)) \|_{L_w^2} \\ &= \left(\int_0^1 \left[\sum_{n=0}^{N} (\tilde{a}_n - a_n) \tilde{J}_n(t) \right]^2 tdt \right)^{\frac{1}{2}} + \left(\int_0^1 \left[\sum_{n=0}^{N} a_n (\tilde{J}_n(t) - J_n(t)) \right]^2 tdt \right)^{\frac{1}{2}} \\ &\leq \left(\int_0^1 \left[\sum_{n=0}^{N} | \tilde{a}_n - a_n |^2 \right] \left[\sum_{n=0}^{N} | \tilde{J}_n(t) |^2 \right] tdt \right)^{\frac{1}{2}} + \left(\int_0^1 \left[\sum_{n=0}^{N} | a_n |^2 \right] \left[\sum_{n=0}^{N} | \tilde{J}_n(t) - J_n(t) |^2 \right] tdt \right)^{\frac{1}{2}} \\ &\leq \| \tilde{A} - A \|_2 \left[\sum_{n=0}^{N} \int_0^1 t \, | \tilde{J}_n(t) |^2 \, dt \right]^{\frac{1}{2}} + \| A \|_2 \left[\sum_{n=0}^{N} \int_0^1 t \, | \tilde{J}_n(t) - J_n(t) |^2 \, dt \right]^{\frac{1}{2}}, \end{aligned} \tag{14}$$

$\| . \|_2$ is 2-norm of vectors. Therefore, by use of orthogonality property of Bessel polynomials, we get

$$
\begin{aligned}
\| \tilde{u}_N(t) - u_N(t) \|_{L^2_w} &\le \| \tilde{A} - A \|_2 \left[\sum_{n=0}^{N} \frac{1}{2} [\tilde{J}_{n+1}(1)]^2 \right]^{\frac{1}{2}} + \| A \|_2 \left[\sum_{n=0}^{N} \int_0^1 \left(\sum_{k=[\frac{N-n}{2}]}^{\infty} \frac{(-1)^k t^{2k+n}}{k!(k+n)! 2^{2k+n}} \right)^2 t dt \right]^{\frac{1}{2}} \\
&\le \| \tilde{A} - A \|_2 \left[\sum_{n=0}^{N} \frac{1}{2} [\tilde{J}_{n+1}(1)]^2 \right]^{\frac{1}{2}} + \| A \|_2 \left[\sum_{n=0}^{N} \int_0^1 \sum_{k=[\frac{N-n}{2}]}^{\infty} \frac{t^{4k+2n+1}}{(k!(k+n)! 2^{2k+n})^2} dt \right]^{\frac{1}{2}} \\
&\le \| \tilde{A} - A \|_2 \left[\sum_{n=0}^{N} \frac{1}{2} [\tilde{J}_{n+1}(1)]^2 \right]^{\frac{1}{2}} \qquad (15) \\
&+ \| A \|_2 \left[\sum_{n=0}^{N} \sum_{k=[\frac{N-n}{2}]}^{\infty} \frac{1}{(k!(k+n)! 2^{2k+n})^2 (4k+2n+2)} dt \right]^{\frac{1}{2}}.
\end{aligned}
$$

According to Eq. (12)-(15), we determine the upper bound of error.

5 NUMERICAL RESULTS

In this section, two examples are given to demonstrate the applicability and accuracy of our methods.

Example 5.1. Consider the fractional pantograph differential equation (Rahimkhani et al., 2017)

$$
D^{\gamma} u(t) = \frac{3}{4} u(t) + u(\frac{1}{2} t) + D^{\gamma_1} u(\frac{1}{2} t) + \frac{1}{2} D^{\gamma} u(\frac{1}{2} t) - t^2 - 2 + 1, \qquad 0 < \gamma_1 < \gamma \le 2, \tag{16}
$$

subject to the initial conditions $u(0) = u'(0) = 0.$ In the case $\gamma_1 = 1, \gamma = 2,$ the exact solution is $u(t) = t^2.$ By applying the proposed method with N = 1 and $\gamma_1 = 1, \gamma = 2,$ we obtain the exact solution. This example considered in other papers, results show that present method more accurate than these methods. Maximum absolute error with $\gamma_1 = 1, \gamma = 2$ on the interval [0,1] for Runge-Kutta method is $5.34 \times 10^{-3},$ the one-Leg θ method is $2.81 \times 10^{-1},$ the variational iteration method is 5.55×10^{-3} and generalized fractional-order Bernoulli wavelet method (FBWM) (Rahimkhani et al., 2017) is 1.39×10^{-13}. From Figure 1. (a), we see that, as γ approaches 2, the numerical solutions converge to the exact solution. So that, the results of Runge-Kutta method, the one-Leg θ method and the variational iteration method are considered in (Rahimkhani et al., 2017).

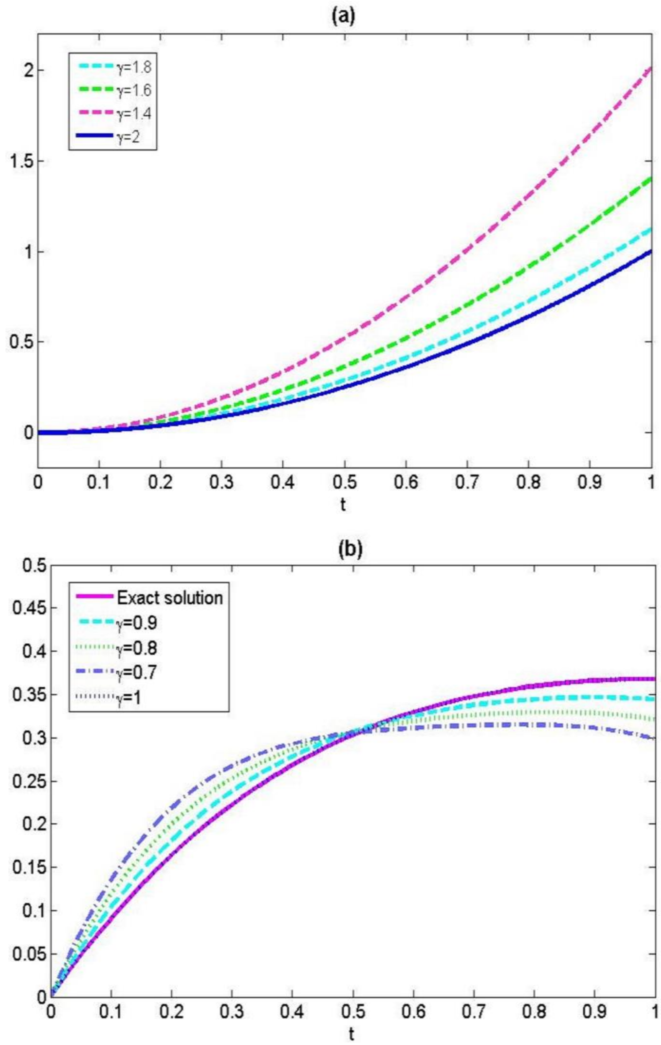

Figure 1: a) Approximation solutions for $\gamma_1 = 1$ and $\gamma = 1.4, 1.6, 1.8, 2,$ with N = 1 of Example 5.1. b) The comparison of $u(t)$ for $\gamma = 0.7, 0.8, 0.9, 1,$ with N = 3 of Example 5.2.

Example 5.2. Consider the fractional pantograph differential equation (Rahimkhani et al., 2017)

$$D^{\gamma}u(t) = -u(t) + 0.1u(\frac{4}{5}t) + 0.5D^{\gamma}u(\frac{4}{5}t) + (0.32t - 0.5)\exp(-0.8t) + \exp(-t), \quad 0 < \gamma \leq 1, \quad (17)$$

subject to the initial condition $u(0) = 0.$ In the case $\gamma = 1,$ the exact solution is $u(t) = t\exp(-t).$ In Table 1, we compare the absolute errors of the proposed method for $\gamma = 1$ with method in (Rahimkhani et al., 2017), variational iteration method and Runge-Kutta method. From Figure 1. (b), we see that, as γ approaches 1, the numerical solutions converge to the exact solution.

Table 1 Absolute error with different values of N with $\gamma = 1$ for Example 5.2.

t	Present method N=3	N=6	N=8	FBWM K=2, M=6	Variational iteration method	Runge-Kutta method
0.1	2.22×10^{-4}	5.46×10^{-8}	2.30×10^{-10}	4.98×10^{-8}	1.30×10^{-3}	8.68×10^{-4}
0.3	1.24×10^{-4}	4.59×10^{-8}	1.10×10^{-8}	7.78×10^{-9}	2.63×10^{-3}	1.90×10^{-3}
0.5	5.65×10^{-5}	1.59×10^{-7}	5.56×10^{-8}	6.34×10^{-5}	2.83×10^{-3}	2.28×10^{-3}
0.7	4.54×10^{-5}	4.66×10^{-7}	1.47×10^{-7}	4.36×10^{-5}	2.39×10^{-3}	2.27×10^{-3}
0.9	8.34×10^{-6}	1.06×10^{-6}	2.84×10^{-7}	2.80×10^{-5}	1.64×10^{-3}	2.03×10^{-3}

6 CONCLUSION

In this work we derive operational matrix of fractional derivative and use it to solve pantograph differential equation of fractional order. Our numerical finding are compared with exact solutions and with the solutions obtained by some other methods. The results of numerical examples demonstrate that this method is more accurate than some existing methods.

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