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Numerical solution of fractional delay differential equations via Fibonnacci polynomials

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ABSTRACT

This paper is concerned with deriving an operational matrix of fractional-order derivative of Fibonacci polynomials. As an application of this matrix, a spectral algorithm for solving some fractional-order initial value problems is exhibited and implemented. The properties of Fibonacci polynomials are presented. The operational matrix of fractional derivative is achieved. This matrix and collocation method are utilized to reduce the solution of the fractional delay differential equations to a system of algebraic equations which can be solved by using Newton's iterative method. Illustrative examples are included to demonstrate the validity and applicability of the technique.

KEYWORDS: Fractional delay differential equation, Fibonacci polynomial, Operational matrix of fractional-order derivative.

1 INTRODUCTION

The fractional calculus may be considered an old and yet new topic. The applications of the delay equation exist in different fields such as number theory, economy, biology, cell growth, etc. there are few works devoted to numerical solution of delay differential equations of fractional order. From these works, we can mention, Legendre collocation method (Saadatmandi et al. 2011), Bernoulli wavelet method (Rahimkhani et al. 2017a), fractional-order Bernoulli wavelet method (Rahimkhani et al. 2017b).
The Fibonacci polynomials and their related polynomials are of interest. For properties and applications of Fibonacci polynomials and some other generalizations, one can be referred to the important book of Koshy (Koshy 2011). Many authors were interested in investigating Fibonacci polynomials and their generalizations from a theoretical point of view (Wang et al. 2015).

Consider the following fractional delay differential equation:

$$\begin{cases} y^{(\alpha)}(x) = f\big(x, y(x), \quad y(x-\tau)\big), & 0 \le x \le 1, \quad n-1 < \alpha \le n, \quad 0 < \tau < 1, \\ y^{(i)}(0) = \gamma_i, & i = 0, 1, \ldots, n-1, \quad n \epsilon N, \end{cases} \tag{1}$$

where, f is an analytical function, τ is delay, γ_i, $i = 0, 1, \dots, n-1$, are real constant and y is the solution to be determined.

2 PRELIMINARIES AND NOTATION

2.1 Properties of fractional calculus

Definition 2.1. The fractional derivative of y in the Caputo sense is defined as (Rahimkhani et al. 2017b)

$$(D^\alpha y)(x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x \frac{y^{(n)}(t)}{(x-t)^{\alpha-n+1}}\, dt, & \alpha > 0, \quad n-1 < \alpha \le n, \\ \dfrac{\partial^n y(x)}{\partial x^n}, & \alpha = n, \end{cases}$$

where n is an integer.

Lemma 2.1. (Rahimkhani et al. 2017b) The Caputo derivative has the following property for $n-1 < \alpha \le n$

$$D^\alpha x^k = \begin{cases} \dfrac{k!}{\Gamma(k-\alpha+1)} x^{k-\alpha}, & k \ge \alpha, \\ 0, & k < \alpha. \end{cases} \tag{2}$$

2.2 Some relevant properties of Fibonacci polynomials

The Fibonacci polynomials can be defined by many ways. For example, they can be generated with the aid of the recurrence relation (Koshy 2011)

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \qquad n \ge 0,$$

with the initial values $F_0(x) = 0$, $F_1(x) = 1$.

They can be generated by the following form representation:

$$F_n(x) = \sum_{j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n-j-1}{j} x^{n-2j-1}.$$

Also, the following inversion formula to above equation in which the polynomials x^i is expressed in terms of the Fibonacci polynomials is of interest

$$x^m = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i [\binom{m}{i} - \binom{m}{i-1}] \, F_{m-2i+1}(x) \, .$$

3 OPERATIONAL MATRIX OF DERIVATIVE

Suppose that the Eq. (1) has a continuous function solution that can be expressed in the Fibonacci polynomials

$$y(x) = \sum_{k=1}^{\infty} a_k F_k(x),$$

and let $y_n(x)$ be an approximation to $y(x)$ that is

$$y(x) \approx y_n(x) = \sum_{k=1}^{m} a_k F_k(x) = A^T F(x), \tag{3}$$

where

$$F(x) = [F_1(x), F_2(x), \ldots, F_m(x)]^T, \qquad A = [a_1, a_2, \ldots, a_m]^T.$$

3.1 Fibonacci operational matrix of fractional derivative

The Caputo derivative of the vector F can be expressed by

$$D^\alpha F(x) = x^{-\alpha} D^{(\alpha)} F(x) \, , \tag{4}$$

where $D^{(\alpha)}$ is the $m \times m$ operational matrix of fractional derivative. In this section, we derive matrix$D^{(\alpha)}$.

Using Eqs. (2), (4) and (5) for $i = 0, 1, \ldots, m-1$,

$$D^\alpha F_{i+1}(x) \;=\; \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} \binom{i-j}{j} D^\alpha x^{i-2j} = \begin{cases} \displaystyle\sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} \binom{i-j}{j} \frac{\Gamma(i+1-2j)}{\Gamma(i+1-2j-\alpha)} x^{i-2j-\alpha}, & i-2j \geq \alpha, \\ 0\,, & i-2j < \alpha. \end{cases}$$

If $i-2j \geq \alpha$, we have

$$D^{\alpha}F_{i+1}(x) = x^{-\alpha}\sum_{j=0}^{\lfloor \frac{i}{2} \rfloor}\binom{i-j}{j}\frac{\Gamma(i+1-2j)}{\Gamma(i+1-2j-\alpha)}x^{i-2j}$$

$$= x^{-\alpha}\sum_{j=0}^{\lfloor \frac{i}{2} \rfloor}\binom{i-j}{j}\frac{\Gamma(i+1-2j)}{\Gamma(i+1-2j-\alpha)}A_{ijk}F_{i-2j+1-2k} = x^{-\alpha}\sum_{j=0}^{\lfloor \frac{i}{2} \rfloor}B_{ijk}F_{i-2j+1-2k},$$

where

$$A_{ijk} = \sum_{k=0}^{\lfloor \frac{i-2j}{2} \rfloor}(-1)^{k}\left[\binom{i-2j}{k}-\binom{i-2j}{k-1}\right], \qquad B_{ijk} = \binom{i-j}{j}\frac{\Gamma(i+1-2j)}{\Gamma(i+1-2j-\alpha)}A_{ijk},$$

then, we have

$$D^{\alpha}F_{i+1}(x) = \begin{cases} x^{-\alpha}\sum_{j=0}^{\lfloor \frac{i}{2} \rfloor}B_{ijk}F_{i-2j+1-2k}, & i-2j \geq \alpha, \\ 0, & i-2j < \alpha. \end{cases} \tag{5}$$

4 NUMERICAL METHOD

In this section, we describe in detail how the operational matrix of fractional derivative of Fibonacci polynomials can be employed for solving fractional delay differential equations.

Substituting approximation obtained into Eq. (1), we have

$$\begin{cases} x^{-\alpha}A^{T}D^{(\alpha)}F(x) = f(x, A^{T}F(x), A^{T}F(x-\tau)), 0 \leq x \leq 1, n-1 < \alpha \leq n, \quad 0 < \tau < 1, \\ A^{T}D^{i}F(0) = \gamma_{i}, \qquad i = 0,1,\dots,n-1, \qquad n \in N. \end{cases} \tag{6}$$

Now, Eq.)6) can be solved by using collocation in nodes $x_p = \frac{p}{m}$, $p = 1,2,\dots,m-n$, where n is the number of conditions of Eq. (1) .

5 ERROR ANALYSIS

Lemma 5.1. Suppose that the function $g:[0,1]\to R$ is m times continuously differentiable, $g\in C^m[0,1]$ and $Y=span\ \{F_1,F_2,\dots,F_m\}$. If A^TF be the best approximation of g out of Y then the mean error bounded is presented as follows:

$$\left\|g-A^TF\right\|_2\le\frac{M}{m!\sqrt{2m+1}},\qquad M=max_{x\in[0,1]}\left|g^{(m)}(x)\right|.$$

Proof. We consider the Taylor polynomial:

$$\tilde{g}(x)=g(0)+g'(0)x+\cdots+g^{(m-1)}(0)\frac{x^{m-1}}{(m-1)!},\qquad |g(x)-\tilde{g}(x)|=\left|g^{(m)}(\eta)\right|\frac{x^m}{m!},$$

where, $\eta\in(0,1)$. Since A^TF is the best approximation g out of Y , $\tilde{g}\in Y$ and using above equation, we have:

$$\begin{aligned}\left\|g-A^TF\right\|_2^2&\le\ \|g-\tilde{g}\|_2^2\\&=\int_0^1|g(x)-\tilde{g}(x)|^2dx=\int_0^1\left[g^{(m)}(\eta)\frac{x^m}{m!}\right]^2dx\le\frac{M^2}{m!^2}\int_0^1x^{2m}dx\\&=\frac{M^2}{m!^2\,(2m+1)},\end{aligned}$$

and by taking square roots we have the above bound.

6 ILLUSTRATIVE TEST PROBLEMS

In this section, we apply the method presented to solve the following two examples.

Example 6.1. Consider the following delay differential equation

$$\begin{cases}y^{(\alpha)}(x)=y(x-\tau)-y(x)+\dfrac{2}{\Gamma(2)}x-1+2\tau x-\tau^2-\tau\,,\qquad 0\le x\le 1,\\ y(0)=0.\end{cases}$$

The exact solution for $\alpha=1$ is $y(x)=x^2-x$.

We applied our numerical method to solve this problem for $m = 3$ in $\tau = 0.01, 0.001$ and we obtain the exact solution in the cases. Table 1 shows the errors of the numerical method via Bernoulli wavelet (Rahimkhani et al. 2017a). From these results, we see that we can achieve a good approximation with the exact solution by using a small number of bases.

Table 1 Absolute errors of Bernoulli wavelet $M = 3, K = 2$ in Example 6.1.

x	$\tau = 0.001$	$\tau = 0.01$
0.2	1.94×10^{-16}	0
0.4	3.33×10^{-16}	1.11×10^{-16}
0.6	8.60×10^{-14}	3.15×10^{-14}
0.8	8.57×10^{-14}	3.23×10^{-14}

Example 6.2. Consider the following fractional delay differential equation:

$$\begin{cases} y^{(\alpha)}(x) = -y(x-0.3) - y(x) + e^{-x+0.3}, & 0 \le x \le 1, 2 < \alpha \le 3 \\ y(0) = 1, y'(0) = -1, y''(0) = 1. \end{cases}$$

The exact solution, when $\alpha = 3$ is $y(x) = e^{-x}$.

Table 2 displays the approximate solutions obtained for various values of x by using the present method with $m = 12$, the Hermit wavelet method (Saeed et al. 2014) for $m = 25$, together with the exact solution.

We get the same results as results of Hermit method using less number of bases.

Table 2 Comparison of the approximate solution with the exact solution in example 6.2.

x	**Exact**	**Our method (m=12)**	**Hermit wavelet (m=25)**
0.2	0.8187	0.8187	0.8187
0.4	0.6703	0.6703	0.6703
0.6	0.5488	0.5488	0.5488
0.8	0.4493	0.4493	0.4493

Figure 1 shows the numerical solutions of example 6.1 and example 6.2 converge to exact solution.

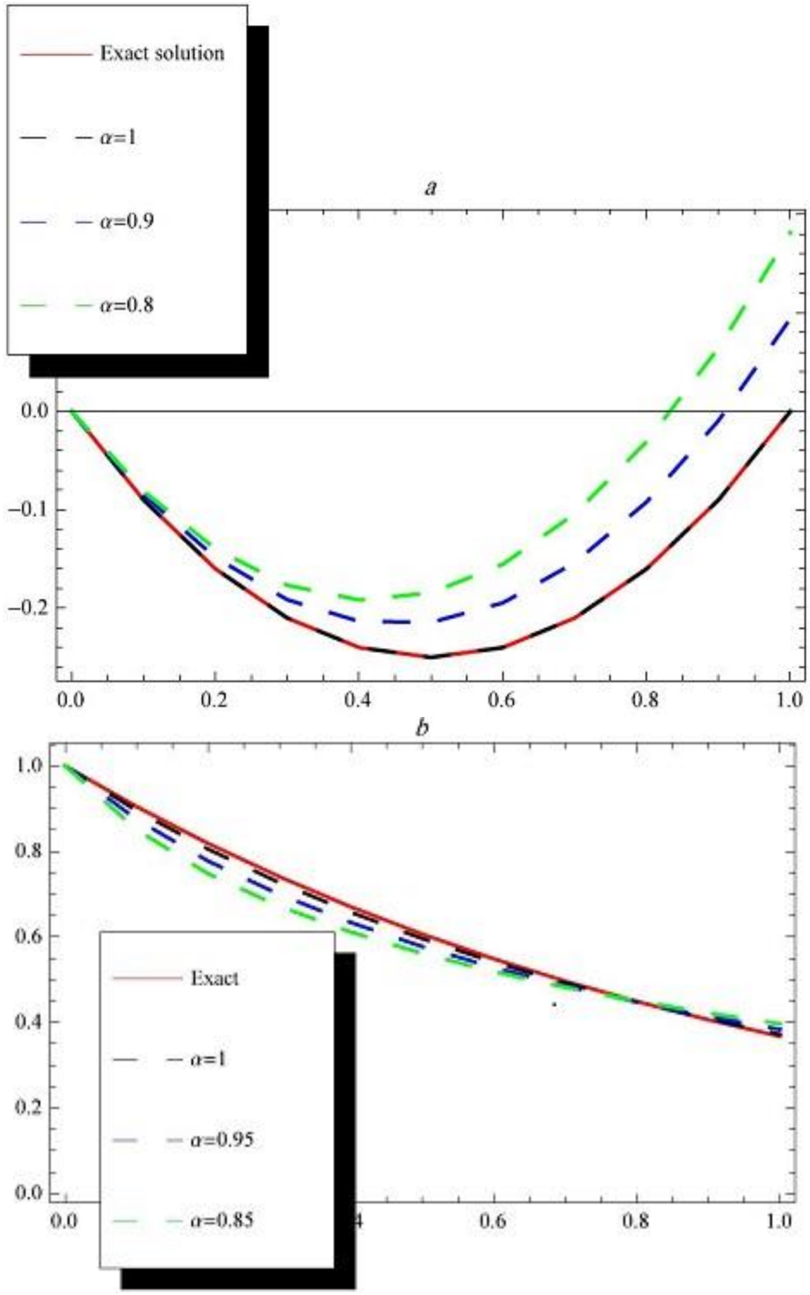

Figure 1: The comparison of approximate solutions for and the exact solution for (a) $n = 2, \tau = 0.001$ for example 6.1, (b) $n = 12$ for example 6.2, with various choices of α.

7 CONCLUSIONS

The aim of present work is to develop an efficient and accurate method for solving fractional delay differential equations. The problem has been reduced to solve a system of algebraic equations which can be solved by using Newton's iterative method. Illustrative examples are included to demonstrate the validity and applicability of this technique.

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