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# Asymptotic Analysis of a Paths Counting Problem

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**ABSTRACT**

In this paper, we give a precise study of the asymptotic behaviour of the total number $P_{m,n}$ of certain restricted lattice paths in an $m \times n$ grid of cells, which appeared in the context of counting train paths through a rail network. Our results are obtained by applying complex analytic techniques, namely the so called diagonalization method and the saddle point method and use the explicit formula for the bivariate generating function of $P_{m,n}$.

**KEYWORDS:** Restricted lattice paths, Asymptotic enumeration, Diagonalization method, Saddle point method.

## 1 INTRODUCTION

Albrecht and White (2008) have studied the problem of counting the number $P_{m,n}$ of lattice paths in an $m \times n$ rectangular grid starting at $(1,p)$ and ending at $(m,q)$, for some $p$, $q$ with $1 \le p \le q \le n$, where the permissible moves from $(i,j)$ are to $(i,j+1)$, $(i+1,j)$ or $(i+1,j+1)$; throughout this paper we speak then about "permissible lattice paths". These numbers arise in connection with a scheduling problem for train paths in a rail network (see Albrecht and White (2008), Panholzer et al. (2012) and Krattenthaler (2015)), where, as the authors mention, the order of magnitude of the numbers $P_{m,n}$ for large values of $m$ and $n$ would be of interest. An example visualizing all such paths for $m = 2$ and $n = 3$ is given in Figure 1.

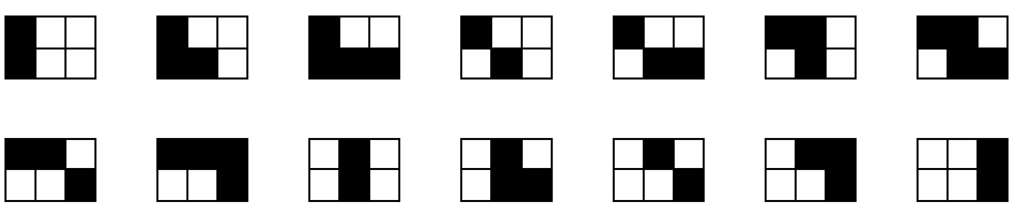

Figure 1: A visualization of all 14 permissible lattice paths for a grid of $m = 2$ rows and $n = 3$ columns; thus $P_{2,3} = 14$.

The initial condition $P_{1,n} = n(n+1)/2$ is clear. Next, consider the following constructive way to see the recurrence for $P_{m,n}$. At the end of each path counted by $P_{m-1,n}$ attach one vertical step. For any path counted by $P_{m-1,j}$ with $j < n$, complete one of the following two constructions:

- attach one vertical step followed by $n - j$ horizontal steps to the end of the original path, or
- attach one diagonal step followed by $n - j - 1$ horizontal steps to the end of the original path.

This construction yields

$$P_{m,n} = P_{m-1,n} + 2\sum_{j=1}^{n-1} P_{m-1,j}, \qquad \text{for } m \geq 2.$$

By solving the above bivariate recurrence via generating functions an explicit formula for $P_{m,n}$ has been obtained in Albrecht's paper. Hirschhorn (2009) obtained the following simpler expression for $P_{m,n}$:

$$P_{m,n} = \sum_{k\geq 0} 2^k \binom{m-1}{k}\binom{n+1}{k+2}. \tag{1}$$

Using this formula (1), Hirschhorn (2009) could describe the asymptotic behaviour of $P_{m,n}$ for the particular case $m = n$, i.e., the diagonal elements, as follows:

$$P_{m,m} \sim \frac{1}{\sqrt{16\pi}\sqrt{2m}}\left(\sqrt{2}+1\right)^{2m+1}, \qquad \text{for } m \to \infty.$$

Based on numerical computations he also made the following conjecture about the second order term in the asymptotic expansion of $P_{m,m}$:

$$P_{m,m} \sim \frac{1}{\sqrt{16\pi}\sqrt{2m}}\left(\sqrt{2}+1\right)^{2m+1} \cdot \left(1 - \frac{c_1}{m} + o\left(m^{-1}\right)\right), \qquad \text{for } m \to \infty, \tag{2}$$

with a constant $c_1 \approx 0.824524$.

For obtaining our results, we use the following explicit formula for the generating function of $P_{m,n}$, which has been computed by Hirschhorn (2009):

$$P(x, z) := \sum_{m,n\geq 1} P_{m,n} x^m z^n = \frac{xz}{(1-z)^2(1-x-z-xz)}. \tag{3}$$

## 2 BRIEF PROOFS OF MAIN RESULTS

In this section, in combination with singularity analysis of generating functions to obtain precise results concerning the asymptotic behaviour of the diagonal elements $P_{m,m}$ and $P_{m,m}$.

Let $r_{m,m}$ be the number of paths a king can cross an $m \times m$ chessboard from a corner of the board to the opposite one, where only the three kind of "forward moves" are permissible; we are now going to make this relation precise. It has been shown by Hirschhorn (2009) that

$$R(t) := \sum_{m\geq 0} r_{m+1,m+1} t^m = \frac{1}{\sqrt{1-6t+t^2}}. \tag{4}$$

**Lemma 1.** *If $\Delta f(m) := f(m+1) - f(m)$ denotes the forward difference operator for all function f, then*

$$P_{m,m} = \frac{1}{8}\Delta^2 r_{m,m}, \qquad \textit{for } m \geq 1. \tag{5}$$

**Proof.** From (3) and by diagonalization method, for $|t|<1/12$, we obtain

$$\hat{P}(t) := \sum_{m\geq 1} P_{m,m} t^m = \frac{1}{2\pi i}\oint_C \frac{P(t/z, z)}{z}\, dz$$

$$= -\operatorname*{Res}_{z=z_1(t)} \frac{t}{(1-z)^2\left(z-z_1(t)\right)\left(z-z_2(t)\right)}$$

$$= -\frac{t}{\left(1-z_1(t)\right)^2\left(z_1(t)-z_2(t)\right)} = \frac{(1-t)^2}{8t\sqrt{1-6t+t^2}} - \frac{1+t}{8t},$$

where $z_1(t)$ and $z_2(t)$ are the solutions of the equation $z^2-(1-t)z+t$ and $C$ is a circle around the origin with radius 3/10. By (4) we imply $8t\hat{P}(t) = (1-t)^2 R(t) - (1+t)$. Extracting coefficients shows (5). □

**Theorem 1.** *The number $r_{m+1,m+1}$ of paths a king can cross an $(m+1)\times(m+1)$ chessboard from a corner of the board to the opposite one, where only the three kind of "forward moves" are permissible admits, for $m\to\infty$, the following asymptotic expansion*:

$$r_{m+1,m+1} = \frac{\left(\sqrt{2}+1\right)^{2m+1}}{2\cdot 2^{\frac{1}{4}}\sqrt{\pi}\sqrt{m}}\cdot\left(1+\frac{3\sqrt{2}-8}{32m}+\frac{113-72\sqrt{2}}{1024m^2}+O\left(m^{-3}\right)\right).$$

**Proof.** Since $1-6t+t^2 = (t-\rho)(t-\bar{\rho}) = (1-\rho t)(1-\bar{\rho}t)$, with $\rho := 3+2\sqrt{2}$ and $\bar{\rho} := 3-2\sqrt{2}$, then we expand $R(t)$ around $t = \bar{\rho}$ and get

$$R(t) = \frac{\rho(1-\rho t)^{-1/2}}{\left(\rho^2-1\right)^{\frac{1}{2}}} - \frac{\rho(1-\rho t)^{1/2}}{2\left(\rho^2-1\right)^{\frac{3}{2}}} + \frac{3\rho(1-\rho t)^{3/2}}{8\left(\rho^2-1\right)^{\frac{5}{2}}} + O\left((1-\rho t)^{\frac{5}{2}}\right). \tag{6}$$

Via extracting coefficients from the binomial series and applying singularity analysis, respectively, we get from (6) immediately

$$r_{m+1,m+1} = \rho^{m+1}\left(\frac{\binom{m-1/2}{m}}{\left(\rho^2-1\right)^{1/2}} + \frac{\binom{m-3/2}{m}}{2\left(\rho^2-1\right)^{3/2}} + \frac{3\binom{m-5/2}{m}}{8\left(\rho^2-1\right)^{5/2}} + O\left(m^{-\frac{7}{2}}\right)\right).$$

To get the result we require an asymptotic expansion of binomial expressions for $m\to\infty$, which can be obtained easily by using Stirling's formula for the factorials. □

Finally, using (5) Theorem 1 leads after easy computations also to the following asymptotic result concerning the numbers $P_{m,m}$.

**Theorem 2.** The *number $P_{m,m}$ of permissible lattice paths in an $m\times m$ square grid admits, for $m\to\infty$, the following asymptotic expansion:*

$$P_{m,m} = \frac{\left(\sqrt{2}+1\right)^{2m+1}}{2\cdot 2^{\frac{1}{4}}\sqrt{\pi}\sqrt{m}}\cdot\left(1-\frac{c_1}{m}+\frac{c_2}{m^2}+O\left(m^{-3}\right)\right),$$

*with* $c_1 = \frac{8+13\sqrt{2}}{32} \approx 0.824524$, *and* $c_2 = \frac{401+312\sqrt{2}}{1024} \approx 0.822494$.

**Theorem 3.** *The number $P_{m,n}$ of permissible lattice paths in an $m \times n$ rectangular grid admits, for $m = \alpha n$, with $\alpha > 0$ fixed, $r = \sqrt{\alpha^2 + 1} - \alpha$, and $n \to \infty$, the following asymptotic equivalent:*

$$P_{m,n} \sim \frac{(r+1)^m}{2\sqrt{\pi}(1-r)^{m+1} r^{n-\frac{1}{2}} \sqrt{r^2+1}\sqrt{m}} = \frac{\sqrt{r}}{2\sqrt{\pi}\sqrt{\alpha}(1-r)\sqrt{r^2+1}\sqrt{n}} \cdot \left( \frac{(r+1)^\alpha}{(1-r)^\alpha r} \right)^n .$$

**Proof.** From (3), we first obtain

$$P_{m,n} = \left[x^m z^n\right] P(x,z) = \left[x^{m-1} z^{n-1}\right] \frac{1}{(1-z)^3 \left(1 - x \frac{1+z}{1-z}\right)} = \left[z^{n-1}\right] \frac{(1+z)^{m-1}}{(1-z)^{m+2}}.$$

Due to Cauchy's integral formula we can write this expression as a contour integral:

$$P_{m,n} = \frac{1}{2\pi i} \oint_C \frac{(1+z)^{m-1}}{z^n (1-z)^{m+2}} dz := \frac{1}{2\pi i} \oint_C g(z)\, dz := I,$$

where $C$ is the punctured unit disc. To evaluate the integral expression $I$ asymptotically, we choose the contour $C$ in such a way that it passes through the saddle point $z = r'$ located on the positive real axis. The saddle point satisfies $g'(z) = 0$. Thus the saddle point of interest, for $m = \alpha n$ and $n \to \infty$, is given by:

$$r' = \frac{-(2m+1) + \sqrt{(2m+1)^2 + 4n(n+3)}}{2(n+3)} = \sqrt{1+\alpha^2} - \alpha + O\left(n^{-1}\right).$$

In the present problem it suffices that the contour $C$ does not really pass through the saddle point $r'$, but just passes by closely. We choose thus as contour $C$ a positively oriented circle around the origin with radius $r := \sqrt{1+\alpha^2} - \alpha$. The idea of the saddle point method is that the main contribution of the contour integral comes from the curve in a small neighbourhood of the saddle point. So we write the integral expression as $I = I_1 + I_2$ where we split the contour $C$ into the following two parts, for $\varphi_0 = n^{-1/2+\varepsilon}$:

$$I = I_1 + I_2 := \frac{1}{2\pi i}\left( \oint_{C_1} g + \oint_{C_2} g \right), \quad C_1 := \left\{ re^{i\varphi} : -\varphi_0 < \varphi < \varphi_0 \right\}, \quad C_2 := \left\{ re^{i\varphi} : \varphi_0 < \varphi < 2\pi - \varphi_0 \right\}.$$

First we evaluate $I_1$, which turns out to give the main contribution of $I$, whereas $I_2$ is asymptotically negligible. From $z = re^{i\varphi} = r + ir\varphi - r\varphi^2/2 + O\left(\varphi^3\right)$ and $rm/(r+1) + rm/(1-r) - n = 0$, we have

$$g(z) = \frac{(1+r)^{m-1}}{r^n (1-r)^{m+2}} \cdot e^{-\frac{r}{2}\left(\frac{1}{(r+1)^2} + \frac{1}{(1-r)^2}\right) m\varphi^2} \cdot \left(1 + O\left(n^{-1/2+3\varepsilon}\right)\right). \tag{7}$$

By using the substitution $\varphi = t/\sqrt{n}$, the equation (7) leads to the following asymptotic evaluation of $I_1$:

$$I_1 = \frac{1}{2\pi i} \oint_{C_1} g(z)\, dz \sim \frac{r(1+r)^{m-1}}{2\pi r^n (1-r)^{m+2}} \int_{-\varphi_0}^{\varphi_0} e^{-\frac{r}{2}\left(\frac{1}{(r+1)^2} + \frac{1}{(1-r)^2}\right) m\varphi^2} d\varphi$$

$$\sim \frac{r(1+r)^{m-1}}{2\pi r^{n-1}(1-r)^{m+2}\sqrt{n}} \int_{-\infty}^{\infty} e^{-\frac{r}{2}\left(\frac{1}{(r+1)^2} + \frac{1}{(1-r)^2}\right)\frac{m}{n} t^2} dt \sim \frac{(1+r)^m}{2\sqrt{\pi} r^{n-1/2} (1-r)^{m+1} \sqrt{r^2+1}\sqrt{m}},$$

for $n \to \infty$. Via standard manipulations, which are omitted here, one can show

$$|g(z)| \le \frac{\left|1 + re^{i\varphi_0}\right|^{m-1}}{r^n \left|1 - re^{i\varphi_0}\right|^{m+2}} = O\left(\frac{1}{r^n}\left|\frac{1 + re^{i\varphi_0}}{1 - re^{i\varphi_0}}\right|^m\right) = O\left(\frac{1}{r^n}\frac{(1+r)^m}{(1-r)^m} \cdot e^{-m\varphi_0^2 \frac{r(1+r^2)}{(1-r^2)^2}}\right).$$

With the constant $\beta = \alpha r\left(1 + r^2\right) / \left(1 + r^2\right)^2$ , therefore it also holds:

$$|I_2| = \left|\frac{1}{2\pi i}\oint_{C_2} g(z)\, dz\right| = O\left(\frac{(1+r)^m}{r^n (1-r)^m} e^{-\beta n^{2\varepsilon}}\right),$$

which is exponentially small compared to $I_1$. Thus we get $P_{m,n} = I_1 + I_2 \sim I_1$, which proves the theorem. □

## 3 CONCLUSION

The aim of this paper is to give a more detailed study of the asymptotic behavior of the total $P_{m,n}$ of paths in an $m \times n$ grid of cells for large values of *m* and *n*, as it is of interest here. First we demonstrate how to prove the conjecture (2) for the diagonal elements $P_{m,m}$. Secondly, we provide results for the asymptotic behavior of $P_{m,n}$ for $m, n \to \infty$, if $m = \alpha n$ with a positive constant $\alpha \in \mathbb{R}$ , which covers the most important growth range of *m* and *n*, via the saddle point method. We also show that there are relations to the problem of counting the number $r_{\mathrm{m,m}}$ of paths a king can cross an $m \times m$ chessboard from the lower left hand corner to the upper right hand corner by using only moves to a neighbouring square either to the right or upwards or diagonally upwards to the right, which has been studied by Hirschhorn (2000). As a consequence of our computations we also get a refinement on the corresponding asymptotic results stated in Hirschhorn (2000).

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