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Numerical solution of parabolic and hyperbolic partial differential equations with nonlocal conditions

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ABSTRACT

The partial differential equation with an integral condition in one, two or three space dimensions, arises in many physical phenomena. In this paper, we propose a numerical scheme to solve parabolic and hyperbolic equations with classical and integral boundary conditions using collocation points and approximating the solution using radial basis functions (RBFs). This method will be used to reduce the problem to a set of algebraic equations. The results of numerical experiments are presented, and are compared with the results of other methods to confirm the validity and applicability of the presented scheme.

KEYWORDS: Radial basis functions; Parabolic equation; Hyperbolicl equation; Integral boundary conditions.

1 INTRODUCTION

Various problems arising in chemistry and physics can be modeled by nonlocal initial-boundary value problems with integral boundary conditions. However, these PDEs subject to nonclassical conditions were analyzed by mathematicians, but improvements of the existing techniques should be done to get more precise solutions. This class of boundary value problems has been attentioned in [2,8] for parabolic and in [1,3,5] for hyperbolic partial differential equations. It is important to be able to convert non-local boundary value problems to a more desirable form such as Bernstein Tau technique [10], Legendre collocation method [9], Crank-Nicolson scheme [8], parallel algorithm [2], non-polynomial spline method [1], optimal explicit method [3] and radial basis function based finite difference method [5].

Consider the following non-classic parabolic equation:

$$u_t(x,t)=u_{xx}(x,t)+Q(x,t)\ , \qquad 0<x<1, 0<t\le 1, \tag{1}$$

with the initial condition:

$$u(x,0)=f(x)\ , \qquad 0\le x\le 1, \tag{2}$$

and nonlocal boundary conditions:

$$\lambda_1 u(0,t) = \int_0^1 k_1(x,t)u(x,t)\,dx + g_1(t), \qquad 0 < t \le 1, \tag{3}$$

$$\lambda_2 u(1,t) = \int_0^1 k_2(x,t)u(x,t)\,dx + g_2(t), \qquad 0 < t \le 1. \tag{4}$$

Also consider the following hyperbolic equation:

$$u_{tt}(x,t) = u_{xx}(x,t) + Q(x,t), \qquad 0 < x < 1, 0 < t \le 1, \tag{5}$$

with the initial conditions:

$$u(x,0) = f(x), \qquad u_t(x,0) = h(x), \qquad 0 \le x \le 1, \tag{6}$$

and the nonlocal boundary conditions of integral type (3) and (4) where $Q(x,t), f(x), h(x), \lambda_1$, $\lambda_2, k_1(x,t), k_2(x,t), g_1(t)$ *a*nd $g_2(t)$ r known functions, while the function u should be determined.

2 RADIAL BASIS FUNCTION

Radial basis functions (RBFs), in irregular domains or higher dimensional geometry, have attracted attention of analysts in science. Commonly used types of these functions include $(r = \| \mathrm{x} - \mathrm{x}_i \|_2)$:

- Multiquadric(MQ): $\phi(r) = \sqrt{\varepsilon^2 + r^2}$
- Inverse Quadratic(IQ): $\phi(r) = \dfrac{1}{(\varepsilon^2 + r^2)}$
- Inverse Multiquadric(IMQ): $\phi(r) = \dfrac{1}{\sqrt{\varepsilon^2 + r^2}}$
- Gaussian(GA): $\phi(r) = e^{-\varepsilon^2 r^2}$

Where ε is a free positive parameter, often referred to as the shape parameter. The optimal choice of the shape parameter is still an open question and it is most often selected by brute force. We remark that the RBFs method has received more and more attention in recent years for solving problems [6].

2.1 Definition of RBF

Let $R^+ = \{x \in R, x \ge 0\}, \|.\|_2$ denotes the Euclidean norm and $\varphi : R^+ \to R$ be a continuous function with $\varphi(0) \ge 0$. A radial basis function on R^d is a function of the form:

$$\phi_i(x) = \varphi(\| \mathrm{x} - \mathrm{x}_i \|_2),$$

which depends only on the distance between $\mathrm{x} \in R^d$ and a fixed point $\mathrm{x}_i \in R^d$. So that the radial basis function ϕ_i is radially symmetric about the center x_i. Let r be the Euclidean distance between a fixed point $\mathrm{x}_i \in R^d$ and $\mathrm{x} \in R^d$, i.e. $\| \mathrm{x} - \mathrm{x}_i \|_2$, [4].

2.2 Numerical procedure

Let $X = L^2([0,1]\times[0,1])$ and

$$\{\psi_{00}(x,t),...,\psi_{0N}(x,t),\psi_{10}(x,t),...,\psi_{1N}(x,t),...,\psi_{N0}(x,t),...,\psi_{NN}(x,t)\}\subset X$$

be the set of Gaussian radial basis functions where $\psi_{ij}(x,t) = e^{-\varepsilon^2((x-x_i)^2+(t-t_j)^2)}$ and

$$H = span\{\psi_{00}(x,t),...,\psi_{0N}(x,t),\psi_{10}(x,t),...,\psi_{1N}(x,t),...,\psi_{N0}(x,t),...,\psi_{NN}(x,t)\},$$

suppose that h be an arbitrary element in X. Since H is a finite dimensional vector space, h has the unique best approximation out of H as $h_{NN}\in H$, that is [7]:

$$\forall g \in H, \| h-h_{NN}\|_2 \le \| h-g\|_2 .$$

Since $h_{NN}\in H$, there exist unique coefficients $v_{00},...,v_{0N},v_{10},...,v_{1N},...,v_{N0},...,v_{NN}$ such that:

$$h \simeq h_{NN} = \sum_{i=0}^{N}\sum_{j=0}^{N} v_{ij}\psi_{ij}(x,t) = V^T\Psi_{NN}(x,t) = \Psi_{NN}^T(x,t)V,$$

where V and $\Psi_{NN}(x,t)$ are vectors with the form:

$$V = [v_{00},...,v_{0N},v_{10},...,v_{1N},...,v_{N0},...,v_{NN}]^T, \tag{7}$$

$$\Psi_{NN}(x,t) = [\psi_{00}(x,t),...,\psi_{0N}(x,t),\psi_{10}(x,t)...,\psi_{NN}(x,t)]^T. \tag{8}$$

In the rest of this section we discuss the application of the radial basis functions for solving nonlocal equation. Let $x_i = \dfrac{i}{N}, i=0,1,...,N; t_j = \dfrac{j}{N}, j=0,1,...,N.$ The solution of the problem is considered as:

$$u = \sum_{i=0}^{N}\sum_{j=0}^{N} v_{ij}\psi_{ij}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} v_{ij}\, e^{-\varepsilon^2((x-x_i)^2+(t-t_j)^2)}, \tag{9}$$

where $\psi_{ij}(x,t)$ is the GA-RBF and v_{ij} are unknown which remain to be determined. By using Eq. (9) in (1)-(6), the collocation technique is used for finding unknown v_{ij}. We collocate (1) in $(N-1)\times N$ interior points $\{(x_l,t_k)\,|\, l=1,...,N-1, k=1,...,N\}$, we get:

$$u_t(x_l,t_k) = u_{xx}(x_l,t_k) + Q(x_l,t_k). \tag{10}$$

Now, collocating (2) in $N-1$ points $x_l, l=1,2,...,N-1$, we obtain:

$$u(x_l,0) = f(x_l). \tag{11}$$

By collocating (3) and (4) in $N+1$ points $t_k, k=0,1,2,...,N$, we have:

$$\lambda_1 u(0,t_k) = \left(\int_0^1 k_1(x,t)u(x,t)\,dx\right)\Big|_{t=t_k} + g_1(t_k), \tag{12}$$

$$\lambda_2 u(1,t_k) = \left(\int_0^1 k_2(x,t)u(x,t)\,dx\right)\Big|_{t=t_k} + g_2(t_k). \tag{13}$$

Also we can collocate (5) in $(N-1)\times(N-1)$ interior points $\{(x_l,t_k)\,|\, l=1,...,N-1, k=2,...,N\}$, we get:

$$u_{tt}(x_l,t_k) = u_{xx}(x_l,t_k) + Q(x_l,t_k). \tag{14}$$

Collocating (6) in $N-1$ points $x_l, l=1,2,...,N-1$, we obtain:

$$u(x_l,0) = f(x_l), \qquad u_t(x_l,0) = h(x_l), \qquad 0\le x\le 1. \tag{15}$$

Eqs. (10)-(13) or (14),(15),(12) and (13) give an $(N+1)\times(N+1)$ system of linear equations, which can be solved for $v_{ij}, i=0,1,...,N, j=0,1,...,N.$ So the approximate solution of problem (1)-(6) can be found.

3 NUMERICAL EXAMPLES

In this section we give some computational results of numerical experiments with the method described in the preceding sections, to support our theoretical discussion.

Example 1. Consider the nonlocal initial-boundary value problem [10,9]

$$\begin{cases} u_t - u_{xx} = (\pi^2+1)e^t \sin(\pi x), & 0 \le x \le 1, 0 \le t \le 1 \\ u(x,0) = \sin(\pi x), & \\ u(0,t) = 0, \quad u(1,t) = 0. & \end{cases} \tag{16}$$

The exact solution of this problem is

$$u(x,t) = e^t \sin(\pi x). \tag{17}$$

In Table 1 we give the absolute errors for GA radial basis function method with $dx = dt = 0.083$ and $\varepsilon = 0.1$ and $dx = dt = 0.062$ with $\varepsilon = 0.3$. From this table one can see the efficiency of the proposed scheme with respect to the Bernstein Tau method of [10] and Legendre collocation method of [9]. The exact and estimated solutions at $x = 0.25$ are given in Fig. 1.

Table 1: Absolute values of error for u from Example 1.

(x,t)	Method [10]	Method [9]	present method with	
			$N=12, \varepsilon=0.1$	$N=16, \varepsilon=0.3$
(0.1,0.1)	$2.31e-06$	$7.87e-05$	$1.89e-07$	$3.55e-13$
(0.2,0.2)	$2.52e-06$	$2.05e-04$	$9.28e-07$	$2.54e-13$
(0.3,0.3)	$2.87e-06$	$3.11e-04$	$6.05e-07$	$1.31e-13$
(0.4,0.4)	$3.20e-06$	$3.79e-04$	$6.50e-07$	$5.78e-14$
(0.5,0.5)	$3.56e-06$	$4.03e-05$	$7.07e-08$	$2.11e-14$
(0.6,0.6)	$3.97e-06$	$3.85e-04$	$2.81e-07$	$7.98e-15$
(0.7,0.7)	$4.47e-06$	$3.28e-04$	$5.27e-07$	$2.86e-15$
(0.8,0.8)	$5.02e-06$	$2.38e-04$	$6.64e-07$	$2.92e-15$
(0.9,0.9)	$6.08e-06$	$1.2e-04$	$2.74e-07$	$7.98e-16$
(1,1)	$7.63e-30$	$1.32e-07$	$3.00e-07$	$3.14e-16$

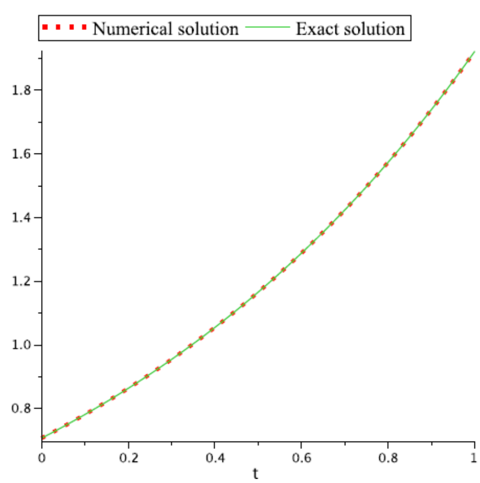

Figure. 1. Analytical and estimated solutions with $dx = dt = 0.083$ and $\varepsilon = 0.1$ at $x = 0.25$ for Example 1.

Example 2. Consider the nonlocal initial-boundary value problem [2,8]

$$\begin{cases} u_t - u_{xx} = (\pi^2 - 1) e^{-t} (\sin(\pi x) + \cos(\pi x)), & 0 \le x \le 1, 0 \le t \le 1 \\ u(x,0) = \sin(\pi x) + \cos(\pi x), & \\ \int_0^1 2\sin(\pi x) u(x,t)\, dx = 0, \qquad \int_0^1 -2\cos(\pi x) u(x,t)\, dx = 0. & \end{cases} \tag{18}$$

The exact solution of this problem is

$$u(x,t) = e^{-t} (\sin(\pi x) + \cos(\pi x)). \tag{19}$$

In Table 2 we give the absolute errors at $x = 0.25$ for GA radial basis function method with $dx = dt = 0.1$ and $\varepsilon = 0.6$ and $dx = dt = 0.062$ with $\varepsilon = 1.1$. We compared our result together with the absolute errors for Crank-Nicolson scheme [8] and parallel algorithm [2]. The exact and estimated solutions at $x = 0.25$ are given in Fig. 2.

Table 2: Absolute values of error for u from Example 2.

t	Method [8]	Method [2]	present method with	
			$N = 12, \varepsilon = 0.1$	$N = 16, \varepsilon = 0.3$
0.1	$5.17e-05$	$4.8e-06$	$6.8e-09$	$1.73e-11$
0.2	$6.19e-05$	$4.7e-06$	$4.75e-09$	$4.55e-12$
0.3	$6.49e-05$	$3.9e-06$	$4.30e-09$	$3.11e-13$
0.4	$6.45e-05$	$4.8e-06$	$3.98e-09$	$1.96e-12$
0.5	$6.21e-05$	$5.3e-06$	$3.63e-09$	$2.41e-12$
0.6	$5.64e-05$	$3.7e-06$	$3.28e-09$	$2.41e-12$
0.7	$4.99e-05$	$2.3e-06$	$2.99e-09$	$2.27e-12$
0.8	$4.49e-05$	$1.6e-06$	$2.68e-09$	$2.08e-12$
0.9	$4.08e-05$	$1.1e-06$	$2.50e-09$	$1.89e-12$
1	$3.64e-05$	$1.0e-06$	$1.81e-09$	$2.25e-12$

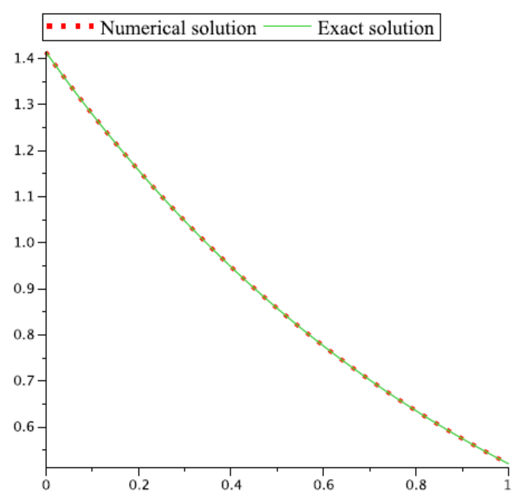

Figure. 2. Analytical and estimated solutions with $dx = dt = 0.1$ and $\varepsilon = 0.6$ at $x = 0.25$ for Example 2.

Example 3. Consider the nonlocal initial-boundary value problem [1,3,5]

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 \le x \le 1, 0 \le t \le 1 \\ u(x,0) = x^2, & u_t(x,0) = \pi\cos(\pi x), \\ u(0,t) = \sin(\pi t), & u(1,t) = \int_0^1 u(x,t)\,dx. \end{cases} \tag{20}$$

The exact solution of this problem is

$$u(x,t) = \cos(\pi x)\sin(\pi t). \tag{21}$$

In Table 3 we give the absolute errors at $t = 0.5$ for Gaussian radial basis function with $dx = dt = 0.090$ and $dx = dt = 0.062$ with $\varepsilon = 0.4$. We compared our method together with non-polynomial spline method [1], optimal explicit method [3] and radial basis function based finite difference method [5]. The exact and estimated solutions at $t = 0.5$ are given in Fig. 3.

Table 3: Absolute values of error for u from Example 3.

x	Method [1]	Method [3]	Method [5]	present method with	
				$N=11, \varepsilon=0.4$	$N=16, \varepsilon=0.4$
0.1	$7.3e-07$	$3.3e-05$	$2.61e-06$	$2.03e-09$	$4.03e-15$
0.2	$1.1e-06$	$3.0e-05$	$3.99e-08$	$2.08e-09$	$1.60e-15$
0.3	$1.0e-06$	$3.2e-05$	$4.61e-06$	$7.93e-10$	$2.99e-15$
0.4	$6.6e-07$	$3.1e-05$	$6.65e-06$	$1.73e-10$	$5.89e-15$
0.5	$7.9e-13$	$3.3e-05$	$4.72e-12$	$1.30e-10$	$2.50e-15$
0.6	$5.6e-07$	$3.4e-05$	$6.65e-06$	$5.45e-11$	$3.66e-15$
0.7	$1.0e-06$	$3.1e-05$	$4.61e-06$	$6.54e-10$	$5.66e-15$
0.8	$1.1e-06$	$3.2e-05$	$3.99e-08$	$1.87e-09$	$9.64e-16$
0.9	$8.5e-07$	$3.4e-05$	$2.61e-06$	$2.03e-09$	$4.33e-15$
1.0	$9.9e-08$	$3.2e-05$	$1.83e-10$	$2.05e-10$	$5.97e-15$

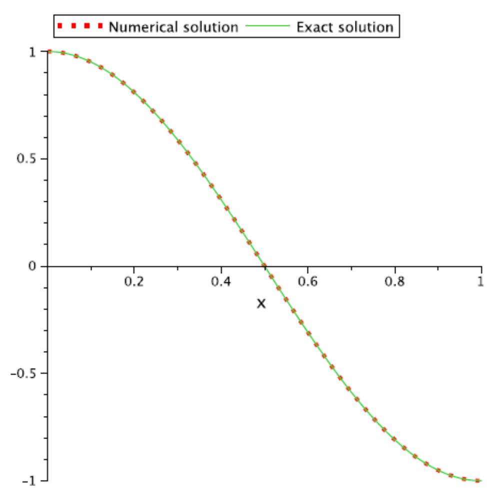

Figure. 3. Analytical and estimated solutions with $dx = dt = 0.090$ and $\varepsilon = 0.4$ at $t = 0.5$ for Example 3.

Example 4. Consider the nonlocal initial-boundary value problem [5]

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 \le x \le 1, 0 \le t \le 1 \\ u(x,0) = \cos(\pi x), & u_t(x,0) = 0, \\ u(0,t) = \cos(\pi t), & \int_0^1 u(x,t)\,dx = 0. \end{cases} \tag{22}$$

The exact solution of this problem is

$$u(x,t) = \cos(\pi x)\cos(\pi t). \tag{23}$$

In Table 4 we give the absolute errors at $t = 0.0001$ for Gaussian radial basis function with $dx = dt = 0.125$ and $dx = dt = 0.067$ with $\varepsilon = 0.8$. We compared our result together with the absolute errors for radial basis function based finite difference method [5]. The exact and estimated solutions at $t = 0.0001$ are given in Fig. 4.

Table 4: Absolute values of error for u from Example 4.

x	Method [5]			present method with	
	MQ-RBF	IMQ-RBF	GA-RBF	$N = 8, \varepsilon = 0.8$	$N = 15, \varepsilon = 0.8$
0.1	$8.18e-06$	$7.66e-06$	$1.61e-05$	$1.43e-07$	$3.82e-16$
0.2	$1.17e-05$	$1.09e-05$	$2.29e-05$	$7.55e-08$	$1.06e-16$
0.3	$1.07e-05$	$1.00e-05$	$2.10e-05$	$3.62e-08$	$1.60e-16$
0.4	$6.29e-06$	$5.91e-06$	$1.24e-05$	$1.48e-08$	$4.7e-17$
0.5	$1.27e-09$	$5.33e-11$	$1.37e-10$	$1.80e-12$	$7.90e-18$
0.6	$6.29e-06$	$5.91e-06$	$1.24e-05$	$1.48e-08$	$1.60e-16$
0.7	$1.07e-05$	$1.00e-05$	$2.10e-05$	$3.62e-08$	$9.75e-17$
0.8	$1.17e-05$	$1.09e-05$	$2.29e-05$	$7.55e-08$	$2.70e-16$
0.9	$8.18e-06$	$7.66e-06$	$1.61e-05$	$1.43e-07$	$2.17e-16$
1.0	$1.11e-10$	$6.99e-12$	$2.10e-11$	$2.03e-11$	$7.68e-17$

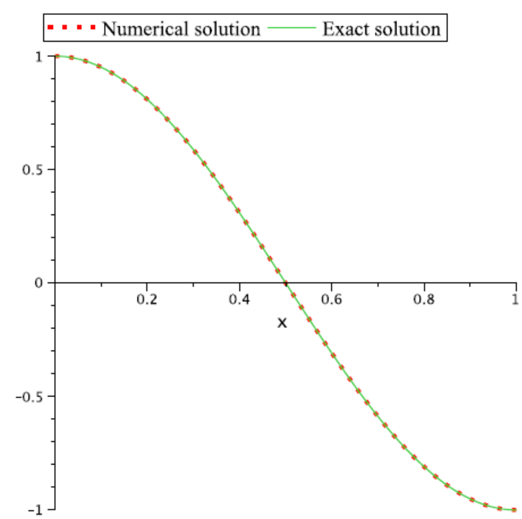

Figure. 4. Analytical and estimated solutions with $dx = dt = 0.125$ and $\varepsilon = 0.8$ at $t = 0.0001$ for Example 4.

4 CONCLUSION

A RBF-based numerical method has been proposed for the solution of two space dimensional linear parabolic and hyperbolic equations subject to appropriate initial and nonlocal boundary conditions. This numerical method uses collocation points and approximates the solution using GA-RBFs. The obtained results showed that this approach using GA-RBFs can solve the problem effectively.

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