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Erratum to on the total graph of a finite commutative ring

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ABSTRACT

In 2012, Shekarriz et al. studied the total graph of a finite commutative ring and found the necessary and sufficient conditions for $\tau(R) \cong C(R)$, but they have errors in its proof. In this paper, we examine this proof and reveal its errors.

KEYWORDS: Total graph, Cayley graph, Maximal cliques.

1 INTRODUCTION

Let R be a commutative ring with nonzero identity, R^+ its additive group, $Z(R)$ its set of zero-divisors and $Cay(R^+, Z(R)\setminus 0)$, which is denoted by $C(R)$, its Cayley graph. The total graph of R was introduced by Anderson and Badawi in [2], as the graph with all elements of R as vertices, and two distinct vertices $x, y \in R$ are adjacent if and only if $x + y \in Z(R)$. Afterwards, in [3], [5] and [6], the authors determined some basic properties of total graph and studied $\tau(R)$, where R is a finite commutative ring.

In [1], G. Aalipour and S. Akbari studied the Cayley graph of a commutative ring with respect to its zero-divisors and determined some properties of it, where $C(R)$ is a graph whose vertices are elements of R and in which two distinct vertices x and y are joined by an edge if and only if $x - y \in Z(R)\setminus\{0\}$. But, prior to this, in [4], Shekarriz et al. tried to answer the naturally arising question: Under what conditions on a finite commutative ring R, do we have $\tau(R) \cong C(R)$?

Before addressing the above question, let us remind some well-known facts about commutative rings: If R is an Artinian ring, then either R is local with its maximal ideal $\mathfrak{m}$, or $R \cong R_1 \oplus \cdots \oplus R_k$, where $k \geq 2$ and each R_i is a local ring with maximal ideal $\mathfrak{m}_i$; this decomposition is unique up to permutation of factors, see [4, Theorem 8.7]. Moreover, if R is finite, then every element of R is either a unit or a zero-divisor. Furthermore, if R is also a local ring with maximal ideal $\mathfrak{m}$, then $\mathfrak{m} = Z(R)$, there exists a prime p such that the characteristic of the residue field $R/\mathfrak{m}$ is p, and $|R|$, $|\mathfrak{m}|$, and

$|R/\mathfrak{m}|$ are all powers of p. Moreover, if $R \cong R_1 \oplus \cdots \oplus R_k$, then $(z_1, \cdots, z_k)$ is a zero-divisor in R if and only if there is an integer i with $1 \le i \le k$, such that $z_i \in Z(R_i)$.

The residue field $R_i/\mathfrak{m}_i$, is denoted by F_i, and $|F_i|$, is shown by f_i. A *clique* in a graph Γ is a subset of pairwise adjacent vertices. In this paper, we followed the notations used in the main article [5].

Shekarriz et al. answered the isomorphic question in [5, Theorem 5.2]: Let R be a finite commutative ring, then $\tau(R) \cong C(R)$ if and only if at least one of the following conditions is true: (a) $R \cong R_1 \oplus \cdots \oplus R_k$, where $k \ge 1$ and each R_i is a local ring of an even order; (b) $R \cong R_1 \oplus \cdots \oplus R_k$, where $k \ge 2$ and each R_i is a local ring and $f_1 = 2$. But, they have errors in its proof when they conclude $\tau(R) \not\cong C(R)$, supposed (a) and (b) do not hold for a finite commutative ring R. In what follows, we indicate that counting the number of vertices of a maximal clique of $\tau(R)$ is very complicated in this case. We also show errors underlying their proof.

2 COUNTING THE NUMBER OF VERTICES OF A MAXIMAL CLIQUE IN $\tau(R)$

In this section, an example will be provided to demonstrate defects of proof given in [5, Theorem 5.2], and we investigate the method of proof too. Hereinafter, the equivalence class $Z(R_i) + a_i$, is denoted by $[a_i]$.

Example 2.1. Let $R = \mathrm{F}_4 \oplus \mathrm{F}_4 \oplus \mathbb{Z}_3$ and $(1,1,1), (0,0,-1) \in R$, denoted by 1 and x, respectively. Then $\tau(\mathrm{F}_4 \oplus \mathrm{F}_4 \oplus \mathbb{Z}_3)$ has five maximal cliques, all containing the edge $\{1, x\}$, which are given separately as follows:

(a) Let $c_1 = ([1],[0],\mathbb{Z}_3)$, $c_2 = (\mathrm{F}_4,[0],[-1])$ and $c_3 = ([1],\mathrm{F}_4,[1])$, then $c_1 \cup c_2 \cup c_3$ forms a maximal clique, where
$$|c_1 \cup c_2 \cup c_3| = |c_1| + |c_2| + |c_3| - |c_1 \cap c_2| - |c_1 \cap c_3| - |c_2 \cap c_3| + |c_1 \cap c_2 \cap c_3|$$
$$= 3 + 4 + 4 - 1 - 1 - 0 + 0 = 9.$$
By permuting the first two components, a new maximal clique will be generated: $([0],[1],\mathbb{Z}_3) \cup ([0],\mathrm{F}_4,[-1]) \cup (\mathrm{F}_4,[1],[1])$. Since, in this example, $|R|/f_1 = |R|/f_2$, these two cliques will be equal in size. Moreover, in these maximal cliques, vertices 1 and x are already counted.

(b) Let $c_1 = ([1],\mathrm{F}_4,[1])$ and $c_2 = ([0],\mathrm{F}_4,[-1])$, then $c_1 \cup c_2$ forms a maximal clique, where $|c_1 \cup c_2| = |c_1| + |c_2| - |c_1 \cap c_2| = 4 + 4 - 0 = 8$. By permuting the first two components, a new maximal clique will be generated: $(\mathrm{F}_4,[1],[1]) \cup (\mathrm{F}_4,[0],[-1])$. Since, in this example, $|R|/f_1 = |R|/f_2$, these two cliques will be equal in size. Moreover, in these maximal cliques, vertices 1 and x are already counted.

(c) Let $c_1 = ([1],[0],[0])$, $c_2 = ([0],[1],[0])$, $c_3 = ([1],[1],[1])$ and $c_4 = ([0],[0],[-1])$, then $c_1 \cup c_2 \cup c_3 \cup c_4$ forms a clique of maximal size 4. It should be noted that, the mutual intersection of every pair of c_i's is empty, for $i = 1, \cdots, 4$, and vertices 1 and x are already counted.

Remark 2.2. Let $R = R_1 \oplus R_2 \oplus R_3$ where R_1 and R_2 are even such that $R_i / Z(R_i) \cong \mathrm{F}_{2^t}$, for $i = 1, 2$ and $t \geq 2$, and R_3 is odd. Then the layouts of equivalence classes of maximal cliques containing the edge $\{1, x\}$ are as the above example.

Now, let us return to the main subject concerning the flaws in the proof of [5, Theorem 5.2]. The findings discussed in the proof are well-reasoned until they were going to show that for $i = 1, \cdots, k$, the edge $\{1, x\}$ does not belong to a maximal $(|R|/f_i)$ -clique in $\tau(R)$.

In that proof, it is supposed that $\{y_s \mid s \in S\}$ is a set of elements of R of maximal size which are adjacent to both 1 and x and also to themselves. It is also cited that if $\{y_s \mid s \in S\} \cup \{1, x\}$ forms a clique of maximal size $(|R|/f_i)$, then there must be $1 \leq m_1 < m_2 < \cdots < m_q \leq k; 0 \leq q \leq k$ such that all y_s's belong to

$$(2.1) \qquad R_1 \oplus \cdots \oplus R_{m_1 - 1} \oplus [a_{m_1}] \oplus R_{m_1 + 1} \oplus \cdots \oplus R_{m_q - 1} \oplus [a_{m_q}] \oplus R_{m_q + 1} \oplus \cdots \oplus R_k$$

Now, according to this direct sum and ambiguity in the assumption, y_s's could be chosen in three following ways:

(1) y_s's belong to (2.1) in which a_{m_i} and m_i are fixed for all $i = 1, \cdots, q$. Based on maximal cliques in the example 2.1(a), 2.1(b) and 2.1(c), $\{y_s \mid s \in S\} \cup \{1, x\}$ is not a maximal clique. It shows that the argument can not be true.

(2) y_s's belong to (2.1) in which only m_i are fixed for all $i = 1, \cdots, q$. Now, example 2.1(a) shows that $\{y_s \mid s \in S\} \cup \{1, x\}$ is not a maximal clique.

(3) y_s's belong to (2.1) in such away a_{m_i}, m_i and q can vary. Thus q will be replaced with q_λ in (2.1), for some $\lambda \in \Lambda$ such that $1 \leq q_\lambda \leq k$, and $y_{S_\lambda} = \{y_s \mid s \in S_\lambda\}$'s are contained in the representation (2.1), where $S_\lambda \subseteq S$ such that for all $s \in S_\lambda$, the elements of y_{S_λ} in (2.1) have a fixed representation (i.e. m_{i_λ} and q_λ are fixed). In Example 2.1, y_{S_λ} is the set of vertices of a clique c_i. Based on deduction in [5, Theorem 5.2], $q_\lambda \neq 1$. If $q_\lambda \geq 2$, then $|y_{S_\lambda}| = \dfrac{|R|}{\prod_{i=1}^{q_\lambda} f_{m_i}}$,

and the required number is calculated by $|\bigcup y_{S_\lambda}|$ as in Example 2.1.

The counting method given in [5, Theorem 5.2] implies that the authors have considered either conditions (1) or (2). Moreover, in the proof, where it is supposed that $2 \leq q \leq k$, if $[a_{m_p}] = [-1_{m_p}]$ and $[a_{m_v}] = [-x_{m_v}]$ for some $v \neq p$, $1 \leq p \leq j$ and $j + 1 \leq v \leq k$, then 1 may belong to $\{y_s \mid s \in S\}$. Correspondingly, if $1 \leq v \leq j$ and $j + 1 \leq p \leq k$, then x may belong to $\{y_s \mid s \in S\}$. Therefore, it is generally incorrect to add 2 in counting the total number of vertices of maximal cliques.

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