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Strongly and Nicely Edge Distance-Balanced Graphs

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ABSTRACT

A nonempty graph G is called nicely edge distance-balanced (NEDB), whenever there exists a positive integer γ'_G, such that for any edge say $e = ab$ we have: $m_G a(e) = m_G b(e) = \gamma' G$. Which $m_G a(e)$ denotes the number of edges laying closer to the vertex a than vertex b and $m_G b(e)$ is defined analogously. Also, a nonempty graph G is strongly edge distance-balanced, for every edge say $e = ab$ of G and every $i \geq 0$ the number of edges at distance i from a and at distance $i + 1$ from b is equal to the number of edges at distance $i + 1$ from a and at distance i from b. In this paper, first we study on some properties of strongly edge distance-balanced graphs. Later, we discuss on some operations of graphs and at last by the help of definition of SEDB graph, classify the NEDB graphs with $\gamma'_G = 3$.

KEYWORDS: Graph, Diameter of graph, Strongly distance-balanced graph, Strongly edge distance-balanced graph, Nicely edge distance-balanced graph.

1 INTRODUCTION

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. If $e = ab \in E(G)$, then $d_G(a, b)$ stands for the distance between a and b in G and it means number of vertices which are lying in the shortest path between a and b. Also, consider any two edges in G, say $e = ab$ and $e' = ab$, the distance between e and e' is defined as:
$d'_G(e') = min\{d'_G(e', a), d'_G(e', b)\}$.
The quantities $n_0(e)$, $n_a(e)$ and $n_b(e)$ are defined to be the number of vertices equidistant from a and b, the number of vertices whose closer to vertex a than vertex b and the number of vertices closer to b than a, respectively. Similarly, the quantities $m_0(e)$, $m_a(e)$ and $m_b(e)$ are defined to be the number of edges equidistant from a and b, the number of edges whose closer to vertex a than b and the number of edges closer to b than a, respectively. Let ab be an arbitrary edge of G. Then for any two non-negative integer i, j, we have:
$D'^{i}_{j}(e) = \{e' \in E(G) \mid d'_G(e', a) = i, \; d'_G(e', b) = j\}$.
By the definition of NEDB, $D'^{1}_{1}(e) = \phi$. The triangle inequality implies that only the sets $D'^{i}_{i-1}(e)$, $D'^{i}_{i}(e)$, $D'^{i-1}_{i}(e)$, for each $(2 \leq i \leq d + 1)$ must be nonempty.
Recall the definition of transmission, $T(a)$ of a vertex $u \in V(G)$ is defined as $T(a) = \Sigma_{b \in V(G)} d(a, b)$, see [10]. Also, consider a as an arbitrary vertex in G, defined the edge-transmission $T'(a)$ defined as $T'(a) = \Sigma_{e \in E(G)} d'(e, a)$.
The graph G is called distance-balanced (as brief DB), if for any arbitrary edge $e = ab$ of G, the number of vertices are lying closer to a than to b is equal to the number of vertices which are lying closer to b than to a, [5, 7, 10].
The simple connected G is called strongly distance-balanced (SDB), if for any edge $e = ab$ in G and any arbitrary integer i, the number of vertices at distance i from a and at distance $i + 1$ from b is equal to the number of vertices at distance $i + 1$ from a and at distance i from b, [1, 9].

A nonempty graph G is called nicely distance-balanced (in short form NDB), whenever there exists a positive integer G, such that for any two adjacent vertices a and b in G, there are exactly G vertices of G which are closer to a than to b, and exactly G vertices of G which are closer to b than to a, see [11].

Edge distance-balanced graphs (as brief EDB), are such graphs in which for every edge $e = ab$ the number of edges closer to vertex a than to vertex b is equal to the number of edges closer to b than to a, [12]. In the other hand, one can easily find a graph G as an EDB graph, if and only if:

$$m_G a(e) = m_G b(e), \text{ for any edge } e = ab \in E(G).$$

The simple connected G is called strongly distance-balanced (SDB), if for any edge $e = ab$ in G and any positive integer i, the number of vertices at distance i from a and at distance i+1 from b is equal to the number of vertices at distance i + 1 from a and at distance i from b, [1, 9].

It was shown in [1] that a graph G with diameter d is strongly distance-balanced if and only if $|Si(a)| = |Si(b)|$, for $i \in \{0, 1, 2, \ldots, d\}$ and every edge ab of G, where

$$Si(a) = \{x \in V(G) | d_G(x, a) = i\}.$$

By above definition, if $|D'^{i-1}_i(e)| = |D'^{i}_{i-1}(e)|$, for $i \in \{1, 2, \ldots, d\}$, then G is edge distance-balanced. But the converse is not true.

2 Some properties of SEDB graphs

In this section, we study on some basic properties of strongly edge distance-balanced graphs and try to understand under which conditions we have SEDB graph.

Proposition 2.1 *If G be a connected and strongly edge distance-balanced graph, then G is regular.*

Proof. Let G be a connected strongly edge distance-balanced graph. Let $e = ab \in E(G)$, by definition of SEDB graph we have $|D'^0_1(e')| = |D'^1_0(e)|$. But we know that, $|D'^0_1(e)| = deg(a) - 1$ and $|D'^1_0(e)| = deg(b) - 1$. Thus, $deg(a) = deg(b)$, for any $a, b \in V(G)$. Hence the result.■

Proposition 2.2 *Let G be a graph with diameter d and* $T_i(a) = \{e \in E(G) \mid d_G(e, a) = i\}$. *If G be strongly edge distance-balanced, then* $|T_i(a)| = |T_i(b)|$, *for any edge e = ab in G and for* $i \in \{1, 2, \ldots, G\}$. *The converse holds if G be a regular graph.*

Proof. Let us assume that G is strongly edge distance-balanced and let $ab \in E(G)$. By definition, we have $|D'^i_{i-1}(e)| = |D'_i i^{-1}(e)|$, for $i \in \{1, 2, \ldots, d\}$. Since $|T_i(a)| = |D'^{i-1}_i(e)| + |D'^i_i(e)| + |D'^i_{i-1}(e)|$, and $|Ti(b)| = |D'^{-+1}_i(e)| + |D'^i_i(e)| + |D'^i_{i-1}(e)|$. So $|Ti(a)| = |Ti(b)|$, for $i \in \{1, 2, \ldots, d\}$. For next part, assume that G is regular. Using induction on i, we now show that $|D'^i_{i-1}(e, e')| = |D'^{i-1}_i(e)|$, holds for every edge say $e = ab \in E(G)$, for any $i \in \{1, 2, \ldots, d\}$. If $i = 1$, then $|D'^0_1(e)| = deg(a) - 1$ and $|D'^1_0(e)| = deg(b) - 1$ and G is regular, we have $|D'^1_0(e)| = \backslash D'^0_1(e)|$. By hypothesis of induction we have $|D'^k_{k-1}(e)| = |D'^{k-1}_k(e)|$, for $1 \leq k \leq d$-1. Observe that,

$$|D'^k_{k+1}(\mathrm{e})| = |T_k(a)| - |D'^k_{k-1}(\mathrm{e})| - |D'^k_k(e)|,$$
$$|D'^{k+1}_k(e)| = |T_k(b)| - |D'^{k-1}_k(e)| - |D'^k_k(\mathrm{e})|.$$

By $|D'^k_{k-1}(e)| = |D'^{k-1}_k(e)|$, hence the result. ■

Proposition 2.3 *Let G be a connected graph with diameter d. If G be strongly edge distance-balanced, then G is strongly distance-balanced.*

Proof. Let G be a connected graph which is strongly edge distance-balanced and for any $a, b \in V(G)$ and $i \in \{1, 2, \ldots, d\}$, we define

$A^a_i = \{u \in V(G) | \exists e = (u,v),\ e' = (u, v') \in T_i(a)$ such that $d_G(e, a) = d_G(e', a) = d_G(u, a) = i\}$,

$B^a_i = \{u \in V(G) \mid \exists e = (u, v), e' = (u, v') \in T_i(a)$ such that $d_G(e, a) = d_G(e', a) = i = d_G(u,a)\}$,

$Cai = \{u \in V(G) | \exists! e = (u, v) \in Ti(a)$ such that $d_G(e,a) = d_G(u, a) = i\}$.

If $a \in A^a_i$ or $a \in C^a_i$, then $a \in S_i(a)$ and if $a \in B^a_i$, then $a \in S_{i+1}(a)$. Thus we have

$$|S_i(a)| = |A^a_i| + |B^a_{i-1}| + |C^a_i| - |C^a_i \cap B^a_{i-1}|.$$

Similarly, we have:

$$|S_i(b)| = |A^b_i| + |B^b_{i-1}| + |C^b_i| - |C^b_i \cap B^b_{i-1}\}.$$

Since G is connected strongly edge distance-balanced graph, we have

$|A^a_i| = |A^b_i|$, $|B^a_i| = |B^b_i|$, $|C^a_i| = |C^b_i|$, $|C^a_I \cap B^a_{i-1}| = |C^b_i \cap B^b_{i-1}|$. Thus, $|S_i(a)| = |S_i(b)|$. Hence the result.■

The converse of the above theorem is not true, for example Generalized Petersen graph $GP(7, 2)$ is strongly distance-balanced graph which is not strongly edge distance-balanced graph.

3 SEDB graph and graph products

In this paper, if G and H are two graphs, the vertex set of Cartesian Product of them is $V(G\square H) = V(G) * V(H)$ and $(x, y)(x', y')$ is an edge of $G\square H$, if $x = x'$ and $yy' \epsilon E(H)$ or $xx' \epsilon E(G)$ and $y = y'$.

Proposition 3.1 *Let G and H be strongly edge distance-balanced as well as vertex distance-balanced graphs. Then G□H is strongly edge distance-balanced graph*

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Proof. Let us assume that the below partition of $E(G\square H)$:

$A = \{(a, x)(b, y) \epsilon E(G\square H) \mid ab \epsilon E(G), x = y\}$,

$B = \{(a,x)(b, y) \epsilon E(G\square H) \mid xy \epsilon E(G), a = b\}$.

Again, assume that G and H are strongly vertex and edge distance-balanced graphs and $(a, x)(b, y) \epsilon A$, for any $i \epsilon \{0, 1, 2, \ldots d\}$, in graph $G\square H$ we have,

$|D'^{\,i}_{i-1}((a, x)(b, y))| = |D'^{\,i}_{i-1}(e)| . |E(H)| + |D'^{\,i}_{i-1}(e)| . |V(H)|$,

$|D'^{\,i}_i((a, x)(b, y))| = |D'^{\,i}_i(e)| . |E(H)| + |D'^{\,i}_i(e)| . |V(H)|$,

$|D'^{\,i}_{i+1}((a, x)(b, y))| = |D'^{\,i}_{i+1}(e)| . |E(H)| + |D'^{\,i}_{i+1}(e)| . |V(H)|$.

In same way we have:

$|D'^{\,i-1}_i((a, x)(b, y))| = |D'^{\,i-1}_i(e)| . |E(H)| + |D'^{\,i-1}_i(e)| . |V(H)|$,

$|D'^{\,i}_i((a, x)(b, y))| = |D'^{\,i}_i(e)| . |E(H)| + |D^i_i(e)| . |V(H)|$,

$|D'^{\,i+1}_i((a, x)(b, y))| = |D'^{\,i+1}_i(e)| . |E(H)| + |D'^{\,i+1}_i(e)| : |V(H)|$.

Therefore,

$|T^{(G\square H)}_i(a, x)| = |S_i(a)| . |E(H)| + |T_i(a)| . |V(H)|$,

$|T^{(G\square H)}_i(b, y)| = |S_i(b)| . |E(H)| + |T_i(b)| . |V(H)|$.

Since G is strongly vertex and edge distance-balanced, we have $|S_i(a)| = |S_i(b)|$ and $|T_i(a)| = |T_i(b)|$. Therefore $|T^{(G\square H)}_i(a, x)| = |T^{(G\square H)}_i(b, y)|$.

Similarly, this result is going to be true for any arbitrary edge say $e = (a, x)(b, y)$ in B. Hence the result.■

Let G and H be two graphs. The corona product $G\circ H$ is obtained by taking one copy of G and $|V(G)|$ copies of H, and by joining each vertex of the i-th copy of H to the i-th vertices of G, $i = 1, 2, \ldots, |V(G)|$, [12].

By the help of definition, every strongly edge distance-balanced graph is edge distance-balanced graph. Also, by [11] next result is clear.

Proposition 3.2 *The corona product of any two nontrivial, connected graphs is not strongly edge distance-balanced.*

Theorem 3.1 *If G is connected and has diameter 2, then the following statements are equivalent:*

a) G is edge distance-balanced,

b) G is strongly edge distance-balanced,

c) G is regular.

Proof. Condition (b) and (c) are equivalent for graphs with diameter 2. Since G has diameter 2, for any vertex $b \epsilon V(G)$, we have $D_e(b,G) = |E(G)| - deg(b)$, where $D_e(b,G) = \Sigma_{e\epsilon E(G)} dG(e, b)$. Thus, $deg(a) = deg(b)$ if and only if $D_e(a,G) = D_e(b,G)$. It was proved in 5 that G is edge-distance-balanced if and only if for every $a, b \epsilon V(G)$, $D_e(a,G) = D_e(b,G)$. Thus the equivalent (a) and (c) follows.■

4 Classification

Theorem 4.1 *A graph G is NEDB graph with* $\gamma'_G = 3$*, if and only if it is one of the following graphs:*
ii) the complete bipartite graph $k_{4,4}$*,*
iii) the Johnson graph $J(5, 1) \approx$ *complete graph* K_5*,*
iv) the Generalized Petersen $GP(3, 1) \approx GP(3, 2)$*,*
v) multipartite graph K_{3*2}.

Proof. Let consider all possible cases for $\gamma'_G = 3$. By [Proposition 2.2], $d \leq \gamma'_G = 3$. On the other hand, d can get 0, 1, 2 or 3. The result for $d = 0$ is clear. Now assume other cases:
First case: If $d = 1$, then G is a complete graph, so $\gamma'_G = n-2$, since $\gamma'_G = 3$ which means a complete graph on 5 nodes, so G is K_5, which is congruent to $J(5, 1)$, hence the proof for (iv).
Second case: If $d = 2$ then we can consider two subcases:
Subcase 1: D'^2_2 (e)= ϕ.
First if we assume D'^3_3 (e)= ϕ, then we conclude $\Sigma_{i=2}^{d+1} | D'^i_i (e)| = 0$. By using Proposition 1.1 and $\gamma'_G = 3$, so number of edges in G must be 7. Now, let us consider the cases which may occurs : If $| D'^2_1$ (e)$|=| D'^1_2$(e) $| = 3$, then D'^{i+1}_i (e) or $D'^i_{i+1}(e)$ must be empty. So we have a tree which is not NEDB.
If $| D'^1_2 (e) |=| D'^2_1 (e) | = 2$, since $\gamma'_G = 3$, so $| D'^2_3$(e) $|=| D'^3_2$(e) $| = 1$. This graph is possible when these two edges in D'^2_3 (e) and D'^3_2(e) are adjacent, (O.W. it is contradiction to D'^3_3 (e)= ϕ). By 6 these assumptions, graph is not regular so it cannot be NEDB, which is a contradiction to Proposition 2.3. Here, $| D'^3_3$(e) $| = 1$, if $|D'^1_2$(e) $|=| D'^2_1$(e) $| = 3$ then G is a tree and it is irregular.
If $| D'^1_2$(e) $|=| D'^2_1$(e) $| = 2$ then $| D'^2_3$ (e)$|=| D'^3_2$(e) $| = 1$. Since, $\gamma'_G = 3$, here we have two vertices of degree 3 and the rest 4 have degree 2, which is not regular graph. At last, consider $| D'^2_1$(e) $|=| D'^1_2$(e)$| = 1$, again graph is irregular.
Subcase 2: D'^2_2 (e)= ϕ.
Also, assume D'^3_3 (e) = ϕ. Again, if we consider $| D'^2_1$(e) $|=| D'^1_2$(e) $| = 3$, we get tree, which is contradiction to NEDB. The all remaining cases as above are irregular except when $| D'^2_2$ (e) $| = 2$ and $| D'^2_1$(e) $|=| D'^1_2(e)| = 2$ and $| D'^3_2$(e) $|=| D'^2_3$(e) $| = 1$ then we get $GP(3, 1)$, which satisfies (v).
By same argument, if $| D'^2_1$(e) $|=| D'^1_2$(e) $| = 3$, one can see for only $| D'^2_2$(e) $| = 5$ and $| D'^2_2$(e) $| = 9$, we have 6 and 8 vertices so the graphs are K_{3*2} and $K_{4,4}$, respectively. Hence , (iii) and (iv).
Third case: $d = 3$.
Subcase 1: D'^2_2(e) = ϕ. First, consider D'^3_3(e) = ϕ , so $| D'^1_1$(e) $| = 0$. Suppose $| D'^2_1$(e) $|=|D'^1_2$(e) $| = 1$ and $| D'^3_2$(e) $|=| D'^2_3$(e) $| = 1$. Because $\gamma'_G = 3$ and here $d = 3$ so $| D'^4_3 |=| D'^3_4$(e) $| = 1$, which is C_7. Hence the proof of (i).
By considering same subcases as before, we can observe that all the other cases are irregular, which are not NEDB.
Forth case: $d = 4$.
Same as before, we can consider two subcases: and if $| D'^2_1$ (e)$|=| D'^1_2$ (e)$| = 1$ and $| D'^3_2$ (e)$|=|D'^2_3$ (e) $| = 1$ and $| D'^4_3$ (e) $|=| D'^3_4$(e) $| = 1$, then we have a cycle on 8 nodes. This is the proof of (ii).
1)D'^2_2 (e)= ϕ and 2)D'^2_2 (e)$\neq \phi$. By same argument, the only possible case occurs when D'^2_2 (e)= ϕ. By considering same subcases as before, we can observe that all the other cases are irregular, which are not NEDB.
Forth case: $d = 4$.
Same as before, we can consider two subcases:
1)D'^2_2 (e)= ϕ and 2)D'^2_2 (e)$\neq \phi$.
By same argument, the only possible case occurs when D'^2_2 (e)$\neq \phi$ and if $| D'^2_1$ (e)$|=|D'^1_2$ (e)$| = 1$ and $| D'^3_2 (e)|=| D'^2_3$ (e)$| = 1$ and $| D'^4_3$ (e)$|=| D'^3_4$ (e)$| = 1$, then we have a cycle on 8 nodes.
This is the proof of (ii). ■

REFERENCES

Balakrishnan K.and. Changat M. and Peterin I. and Spacapan S. (2009). Strongly Distance-Balanced Graph and Graphs Products, European Journal of Combinotrics, 30, 1048—1053.
Biggs N.(1993). Algebraic Graph Theory, Cambridge University Press.
Fukuda K. and Handa K. (1993). Antipodal graphs and oriented matroids, Discrete Math. 111,245--256.
Handa K.(1999). Bipartite graphs with balanced (a,b)-partitions, Ars Combin., 51 , 113--119.
Ilic A. and Klavzar S. and Milonovic M. (2004). On Distance-Balanced Graphs, J. Aust. Math. Soc., 76, 39--49.
Imrich W. and Klavzar S. (2000). Product Graph: Structure and Recognition, J. Wiley Sons, New York.
Jerebic J. and Klavzar S. and Rall D.F. (2000). Distance-Balanced Graphs, University of Maribor Kroska Cesta, 160.
Kovac L. and Kutnar K. and Marusic D. and Wilson S. (2012). Classification of Cubic Symmetric Tricirculants, Electron. J. Combin. 1,p 14--24.
Kutnar K. and Malnic A. and Marusic D. and Miklavic S. (2009). The Strongly Distance-Balanced Property of The Generalized Petersen Graphs, Ars Mathematica Contemporanea, 2, 41--47.
Kutnar K. and Malnic A. and Marusic D. and Miklavic S. (2006). Distance -Balanced Graph: Symmetry Conditions, Discrete Mathematics, 306, 1881--1894.
Kutnar K. and Miklavic S. (2014). Nicely Distance-Balanced Graphs, European Journal of Combinotrics, 39, 57--67.

CONCLUSION

The most important conclusion of this paper is the classification of Nicely Edge Distance-Balanced Graphs according to γ'_G =3. By this assumption we can classify the graphs and also by more calculation we can continue this work for $\gamma'_G \geq 4$ in next work.