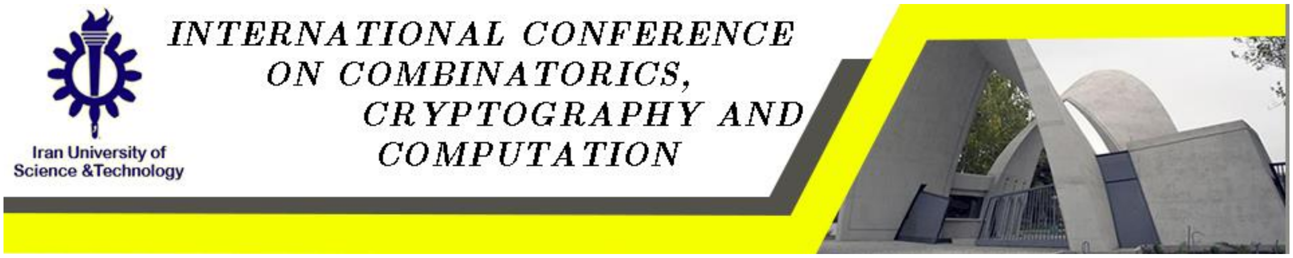

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Total dominator chromatic number in graphs

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ABSTRACT

A given graph G has a total dominator coloring if it has a proper coloring in which each vertex of G is adjacent to every vertex (or all vertices) of some color class. The total dominator chromatic number $\chi_d^t(G)$ of G is the minimum number of color classes in a total dominator coloring of a graph G. In this paper we study the total dominator chromatic number on several classes of graphs.

Keywords: Graph, Total Domination, Coloring,Total dominator coloring.

1 Introduction

For terminology and notation on graph theory not given here, the reader is referred to [10] Throughout this note, all graphs are simple, connected and undirected. Let $G=(V,E)$ be a graph with the vertex V and the edge set E . A *proper coloring* of a graph G is an assignment of colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The *chromatic number* $\chi(G)$ is the minimum number of colors required for a proper coloring of G. A color class is the set of all vertices, having the same color. The color class corresponding to the color i is denoted by V_i. A set $S \subseteq V(G)$ is a *total dominating set* of G if every vertex of $V(G)$ is adjacent to at least one vertex of S . The cardinality of the smallest total dominating set of G, denoted by $\gamma_t(G)$, is called the *total domination number* of G. Total domination is now well studied in graph theory. The literature on the subject on total domination in graphs has been surveyed and detailed in the recent book [3]. A survey of total domination in graphs can also be found in [2]. A dominator coloring, of a graph G is a proper coloring of G such that every vertex of $V(G)$ dominates all vertices of at least one color class. The dominator chromatic number $\chi_d(G)$ of G is the minimum number of color classes in a dominator coloring of G. The concept of dominator coloring was introduced and studied by Gera, Horton and Rasmussen [6] and studied further, for example, by Gera [5, 4] and Chellali and Maffray [1].

A total dominator coloring of a graph G with no isolated vertex is a proper coloring of G in which every vertex of the graph is adjacent to every vertex of some (other) color class. The total dominator chromatic number $\chi_d^t(G)$ of G is the minimum number of color classes in total dominator coloring of G .A $\chi_d^t(G)$ - coloring of G is any total dominator coloring with $\chi_d^t(G)$ colors.

The complement of a graph G is denoted by $\overline{G}$ and is a graph with the vertex set $V(G)$ and for every two vertices v and w, $vw \in E(\overline{G})$ if and only if $vw \notin E(G)$.

A cycle on n vertices is denoted by C_n and a path on n vertices by P_n. A complete graph on n vertices is denoted by K_n. A wheel graph on $n+1$ vertices is denoted by $W_{1,n}$.

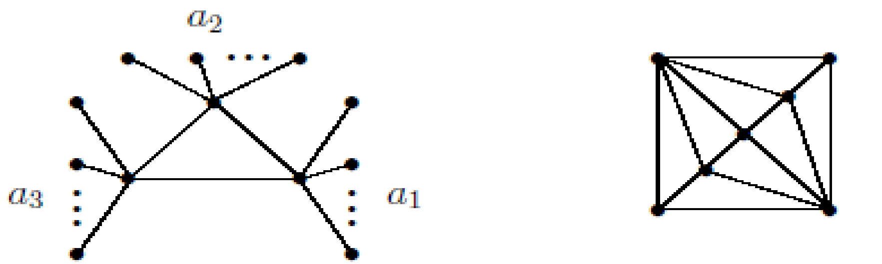
The complete K -partite graph $K_{a_1,a_2,\ldots,a_k}$ is a simple graph whose vertices can be partitioned into sets such that two vertices are adjacent if and only if they are not in the same partite sets.

A complete bipartite graph on $m+n$ vertices is denoted by $K_{m,n}$.

The multi-star graph $K_m(a_1,a_2,...,a_m)$ is formed by joining a_i end-vertices to each vertex x_i of a complete graph K_m for $(1 \le i \le m)$, where $V(K_m) = \{x_1, x_2, ..., x_m\}$.

A double star graph on a_1+a_2+2 vertices is denoted by $K_2(a_1,a_2)$.

The fan graph $F_{m,n}$ is defined as the graph join $\overline{K_m}+P_n$ where $\overline{K_m}$ is the complement of K_m with vertex set $\{u_1,\ldots,u_m\}$ and P_n is a path with vertex set $\{v_1,\ldots,v_n\}$.



Multi-star $K_3(a_1, a_2, a_3)$ Fan graph $F_{2,5}$

Figure 1.

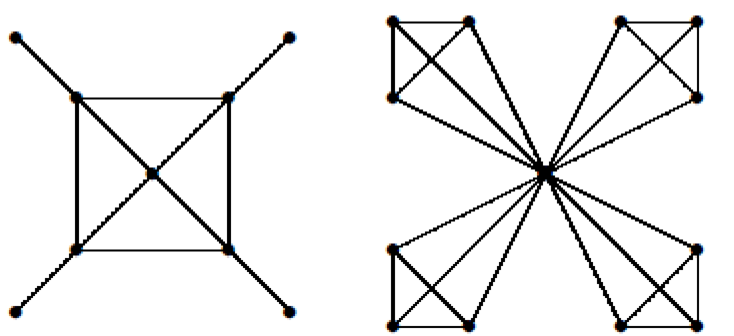
The helm graph H_n is the graph obtained from a wheel graph $W_{1,n}$ by adjoining a pendant edge at each node of the cycle C_n.

A flower graph Fl_n is the graph obtained from a helm by joining each pendant vertex to the central vertex of the helm.

The Sun flower graph Sf_n is the resultant graph obtained from the flower graph of wheels $W_{1,n}$ by adding n pendant edges to the central vertex .

The crown graph S_n^0 for an integer $n \geq 3$ is the graph with vertex set $\{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$ and edge set $\{(u_i, v_j) : 1 \leq i, j \leq n, i \neq j\}$.

The windmill graph $W_n^{(m)}$ is the graph obtained by taking m copies of the complete graph K_n with a vertex in common.



Helm graph H_4 Windmill graph $W_4^{(4)}$

Figure 2.

The n-barbell graph B_n is the simple graph obtained by connecting two copies of a complete graph K_n by a bridge.

The double complete bipartite graph R_2 is obtained by connecting two copies of a complete graph $K_{m,n}$ $(m,n \geq 2)$ by a bridge.

We study total dominator coloring of family of graphs. We need the following theorems .

Theorem 1.1 *[9] Let G be a graph of order n and without isolated vertices. Then*

$$max\{\gamma_t(G), \chi_d(G)\} \leq \chi_d^t(G) \leq n.$$

Theorem 1.2 *[9] If G is a connected graph of order n and without isolated vertices, then $2 \leq \chi_d^t(G) \leq n$. Furthermore, $\chi_d^t(G)$ is 2 or n if and only if G is a complete bipartite graph, or is isomorphic to the complete graph K_n, respectively.*

Theorem 1.3 *[4] (i) The multistar $K_m(a_1, a_2, ..., a_m)$ has $\chi_d(K_m(a_1, a_2, ..., a_m)) = m+1$.*

(ii) The complete k-partite graph $K_{a_1,a_2,...,a_k}$ has $\chi_d(K_{a_1,a_2,...,a_k}) = k.$

Theorem 1.4 *[8] (i) For the helm graph H_n, ($n \geq 3$), $\chi_d(H_n) = n+1$.*

(ii) For the flower graph Fl_n, $\chi_d(Fl_n) = \begin{cases} 3 & if \ \ n \ \ is \ \ even \\ 4 & if \ \ n \ \ is \ \ odd \end{cases}$

(iii) For Sun flower graph Sf_n, $\chi_d(Sf_n) = \begin{cases} 3 & if \ \ n \ \ is \ \ even \\ 4 & if \ \ n \ \ is \ \ odd. \end{cases}$

Theorem 1.5 *[7](i) Let B_n be n-barbel graph. Then $\chi_d(B_n) = n+1$ for $n \geq 3$.*
(ii) Let S_n^0 be a crown graph. Then $\chi_d(S_n^0) = 4$ for $n \geq 3$.

Theorem 1.6 *[9] Let $G = W_n^{(m)}$ be a windmill graph with $n, m \geq 2$, then $\chi_d(G) = n$.*

2 Total dominator chromatic number

The following theorems appeared in [9].
Theorem 2.1 *(i) Let $W_{1,n}$ be a wheel of order $n+1 \geq 4$. Then*

$$\chi_d^t(W_{1,n}) = \begin{cases} 3 & if \;\; n \;\; is \;\; even \\ 4 & if \;\; n \;\; is \;\; odd. \end{cases}$$

(ii) Let C_n be a cycle of order $n \geq 3$. Then

$$\chi_d^t(C_n) = \begin{cases} 2 & if \;\; n = 4 \\ 4\left\lfloor \frac{n}{6} \right\rfloor + r & if \;\; n \neq 4 \; and \;\; for \;\; r = \; 0,1,2,4, \;\; n \equiv r \left(mod \;\; 6\right) \\ 4\left\lfloor \frac{n}{6} \right\rfloor + r - 1 & if \;\; n \equiv r\left(mod \;\; 6\right) \;, \; where \;\; r = 3,5. \end{cases}$$

(iii) Let P_n be a path of order $n \geq 2$. Then

$$\chi_d^t(P_n) = \begin{cases} 2\left\lceil \frac{n}{3} \right\rceil - 1 & if \;\; n = 1 \left(mod \;\; 3\right) \\ 2\left\lceil \frac{n}{3} \right\rceil & otherwise. \end{cases}$$

Now we present the total dominator chromatic number of other classes for completeness in this section. Note that in this graphs $\chi_d^t = \chi_d$.

Theorem 2.2 *(i) The multistar* $K_m(a_1, a_2, ..., a_m)$ *has* $\chi_d^t(K_m(a_1, a_2, ..., a_m)) = m + 1$.

(ii) The complete k-partite graph $K_{a_1, a_2, ..., a_k}$ has $\chi_d^t(K_{a_1, a_2, ..., a_k}) = k$.

(iii) For fan graph $F_{m,n}$, $n \geq 2$, $m \geq 1$ $\chi_d^t(F_{m,n}) = 3$.

(iv) For the helm graph H_n, $n \geq 3$, $\chi_d^t(H_n) = n + 1$.

(v) For the flower graph Fl_n $n \geq$ $\chi_d^t(Fl_n) = \begin{cases} 3 & if \;\; n \;\; is \;\; even \\ 4 & if \;\; n \;\; is \;\; odd. \end{cases}$

(vi) For the sun flower graph Sf_n $n \geq$ $\chi_d^t(Sf_n) = \begin{cases} 3 & if \;\; n \;\; is \;\; ev \\ 4 & if \;\; n \;\; is \;\; od \end{cases}$

(vii) For the barbell graph B_n with $n \geq 3$, $\chi_d^t(B_n) = n + 1$.

(viii) For the crown graph S_n^0 with $n \geq 2$, $\chi_d^t(S_n^0) = 4$.

*(ix)*For the windmill graph $G = W_n^{(m)}$ with $n, m \geq 2$ $\chi_d^t(G) = n$.

(x) Let $\overline{K_2(a_1, a_2)}$ be the complement of a double star $K_2(a_1, a_2)$. Then

$$\chi_d^t\,\overline{(K_2(a_1,a_2)} = \begin{cases} 3 & \text{if } a_1 = a_2 = 1 \\ 4 & \text{if } a_1 = 2, a_2 = 1 \\ a_1 + a_2 & \text{if } a_1, a_2 \geq 3. \end{cases}$$

(xi) The double complete bipartite graph R_2 has $\chi_d^t(R_2) = 4$.

Proof.

(i) Let $K_m(a_1,a_2,...,a_m)$ be the multi-star graph, and let $V(K_m(a_1,a_2,...,a_m))$ = $\{v_1, v_2, \ldots, v_{m+\sum_{i=1}^{m} a_i}\}$ where $\{v_1, v_2, \ldots, v_m\}$ be the vertices of K_m and $\{v_{m+1}, v_{m+2}, ..., v_{m+\sum_{i=1}^{m} a_i}$ } be the end-vertices incident on K_m such that $\{v_{m+1}, v_{m+2}, \ldots, v_{m+a_1}\}$ are adjacent to the vertex v_1 and $\{v_{m+1+\sum_{i=1}^{j-1} a_i}, v_{m+2+\sum_{i=1}^{j-1} a_i}, \ldots, v_{m+\sum_{i=1}^{j} a_i}\}$ are adjacent to the vertex v_j for each $(2 \leq j \leq m)$.

By Theorem 1.3 (i) we have $\chi_d(K_m(a_1,a_2,...,a_m)) = m+1$. Now it follows from Theorem 1.1 that $\chi_d^t(K_m(a_1,a_2,...,a_m)) \geq m+1$. Hence it is sufficient to show that $\chi_d^t(K_m(a_1,a_2,...,a_m)) \leq m+1$. Let $C = \{V_1, V_2, \ldots, V_{m+1}\}$ be the class colorings of $K_m(a_1,a_2,...,a_m)$ in which $V_i = \{v_i\} (1 \leq i \leq m)$ and $V_{m+1} = \{v_{m+1}, v_{m+2}, ..., v_{m+\sum_{i=1}^{m} a_i}\}$. Since v_i is adjacent to all vertices of color class $V_{i'} (1 \leq i, i' \leq m, i \neq i')$ and each vertex $v_k (m+1 \leq k \leq m+a_1)$ is adjacent to all vertices of color class V_1 and for $(2 \leq j \leq m)$ each vertex of vertices $\left\{v_{m+1+\sum_{i=1}^{j-1} a_i}, v_{m+2+\sum_{i=1}^{j-1} a_i}, \ldots, v_{m+\sum_{i=1}^{j} a_i}\right\}$ is adjacent to all vertices of color class V_j, then the coloring C is a χ_d^t- coloring of $K_m(a_1,a_2,...,a_m)$ with $m+1$ colors and $\chi_d^t(K_m(a_1,a_2,...,a_m)) \leq m+1$ Therefore $\chi_d^t(K_m(a_1,a_2,...,a_m)) = m+1$.

(ii) Let $K_{a_1,a_2,\ldots,a_k}$ be a complete k-partite graph and $V(K_{a_1,a_2,\ldots,a_k}) = \{v_1, v_2, \ldots, v_{\sum_{i=1}^{k} a_i}\}$ where $\{v_1, v_2, \ldots, v_{a_1}\}$ be the vertices of first partite set and $\{v_{(\sum_{i=1}^{j-1} a_i)+1}, v_{(\sum_{i=1}^{j-1} a_i)+2}, \ldots, v_{(\sum_{i=1}^{j} a_i)}\}$ be the j-th partite set of $K_{a_1,a_2,\ldots,a_k}$ for every $(2 \leq j \leq k)$. By Theorem ١/٣ (ii) we have $\chi_d(K_{a_1,a_2,\ldots,a_k}) = k$. Now it follows from Theorem ١/١ that $\chi_d^t(K_{a_1,a_2,\ldots,a_k}) \geq k$.

Hence it is sufficient to show that $\chi_d^t(K_{a_1,a_2,\ldots,a_k}) \leq k$. Let $C = \{V_1, V_2, \ldots, V_k\}$ be the coloring of $K_{a_1,a_2,\dots,a_k}$ in which $V_1=\{v_1,v_2\dots,v_{a_1}\}$ and for each $(2\le j\le k)$, $V_j=\{v_{(\sum_{i=1}^{j-1}a_i)+1},v_{(\sum_{i=1}^{j-1}a_i)+2},\dots,v_{(\sum_{i=1}^{j}a_i)}$ 綈 }.

Since each vertex in i-th partite set is adjacent to all vertices of color class V_j for every $(1\le i,j\le k,i\ne j)$, then the coloring C is a χ_d^t-coloring of $K_{a_1,a_2,\dots,a_k}$ with k colors and $\chi_d^t(K_{a_1,a_2,\dots,a_k})\le k$. Therefore $\chi_d^t(K_{a_1,a_2,\dots,a_k})=k.$

(iii) Let $V(F_{m,n})=\{v_1,v_2,\dots,v_{n+m}\}$, where $\{v_1,v_2,\dots,v_n\}$ be the vertices of the path P_n and $\{v_{n+1},v_{n+2},\dots,v_{n+m}\}$ be the vertices of $\overline{K_m}$. Since K_3 is a subgraph of $F_{m,n}$ it follows that at least three colors are needed to color the fan graph. then we have $\chi_d^t(F_{m,n})\ge 3$. Hence it is sufficient to show that $\chi_d^t(F_{m,n})\le 3$. Consider a proper coloring of $F_{m,n}$ in which $V_1=\{v_{n+1},v_{n+2},\dots,v_{n+m}\}$ and when n is odd, $V_2=\{v_1,v_3,\dots,v_n\}$, $V_3=\{v_2,v_4,\dots,v_{n-1}\}$. If n is even, $V_2=\{v_1,v_3,\dots,v_{n-1}\}$ and $V_3=\{v_2,v_4,\dots,v_n\}$,then each vertex of v_i $(n+1\le i\le n+m)$ is adjacent to every vertex of color class V_2 and color class V_3. Also each vertex of v_j $(1\le j\le n)$ is adjacent to every vertex of color class V_1. Therefore this is a total dominator coloring and $\chi_d^t(F_{m,n})\le 3$. Hence $\chi_d^t(F_{m,n})=3$.

(iv) Let $V(H_n)=\{v_1,v_2,\dots,v_{2n+1}\}$, where v_1 is the central vertex, v_i $(2\le i\le n+1)$ be the vertices on the cycle C_n and v_j $(n+2\le j\le 2n+1)$ be the pendant vertices on cycle C_n such that v_{n+i} is adjacent with $v_i(2\le i\le n+1)$. It follows from Theorem ۱/۴ (i) that $\chi_d(H_n)=n+1$. Now by Theorem ۱/۱, we have $\chi_d^t(H_n)\ge n+1$. Hence it is sufficient to show that $\chi_d^t(H_n)\le n+1$. Let $C=\{V_1,V_2,\dots,V_{n+1}\}$ be the coloring of H_n in which $V_1=\{v_1,v_{n+2},v_{n+3}\dots,v_{2n+1}\}$ and $V_i=\{v_i:2\le i\le n+1\}$, then v_{n+i},v_1 are adjacent to all vertices of color class $V_i(2\le i\le n+1)$ and $v_j(2\le j\le n)$ is adjacent to every vertex of color class V_{j+1}. Also v_{n+1} is adjacent to every vertex of color class V_2 and color class V_n. So the coloring C is χ_d^t-coloring and $\chi_d^t(H_n)\le n+1$. Therefore $\chi_d^t(H_n)=n+1$.

(v) By the definition of flower graph, Fl_n is obtained from a helm graph by joining each pendant vertex to the central vertex. We refer the labelling given in the proof (iv). By Theorem ۱/۴ (ii) we have $\chi_d(Fl_n)=3$ if n is even and $\chi_d(Fl_n)=4$ if n is odd. Now it follows from Theorem ۱/۱ that $\chi_d^t(Fl_n)\ge 3$ (n is even) or $\chi_d^t(Fl_n)\ge 4$ (n is odd). Now, it is sufficient to show that $\chi_d^t(Fl_n)\le 3$ or 4 (according as n is even or odd). Suppose n is even and let $C=\{V_1,V_2,V_3\}$ be the coloring of Fl_n in which $V_1=\{v_1\},V_2=\{v_2,v_4,\dots,v_n\}\cup\{v_{n+3},v_{n+5},\dots,v_{2n+1}\},V_3=\{v_3,v_5,\dots,v_{n+1}\}\cup\{v_{n+2},v_{n+4},\dots,v_{2n}\}$. Now if n is odd, let $C=\{V_1,V_2,V_3,V_4\}$ be the coloring of Fl_n in which $V_1=\{v_1\},V_2=\{v_2,v_4,...,v_{n-1}\}\cup\{v_{n+3},v_{n+5},...,v_{2n}\},V_3=\{v_3,v_5,...,v_n\}\cup\{v_{n+2},v_{n+4},...,v_{2n+1}\}$ and $V_4=\{v_{n+1}\}$. Then each vertex $v_i(2\le i\le 2n+1)$ is adjacent to every vertex of color class V_1 and vertex v_1 is adjacent to all vertices of color classes V_2 and V_3, (when n is even) and is adjacent to all vertices of color classes V_2,V_3 and V_4 (when n is odd). Therefore the coloring C is a χ_d^t-coloring of (Fl_n) and hence $\chi_d^t((Fl_n))\le 3$ (when n is even) or $\chi_d^t((Fl_n))\le 4$ (when n is odd).Thus

$$\chi_d^t(Fl_n)=\begin{cases}3 & if\ \ n\ is\ even\\ 4 & if\ \ n\ is\ odd.\end{cases}$$

(vi) By the definition of sun flower graph, Sf_n is obtained from a flower graph by joining each pendant edge to the central vertex. Let $\{v_1,v_2,...,v_{3n+1}\}$, where v_1 is central vertex and v_i $(2\le i\le n+1)$ be the vertices on the cycle C_n. Also for each $(2\le i\le n+1)$, the vertex v_{n+i} is adjacent to v_1,v_i and the pendent vertices v_i for each $(2n+2\le i\le 3n+1)$ is adjacent to vertex v_1 .By Theorem ۱/۴ (iii) we have $\chi_d(Sf_n)=3$ if n is even and $\chi_d(Sf_n)=4$ if n is odd. Now it follows from Theorem ۱/۱ that $\chi_d^t(Sf_n)\ge 3$ (n is even) or $\chi_d^t(Sf_n)\ge 4$ (n is odd). Now, it is sufficient to show that $\chi_d^t(Sf_n)\le 3$ or 4 (according as n is even or odd). Suppose n is even and let $C=\{V_1,V_2,V_3\}$ be the coloring of Sf_n in which $V_1=\{v_1\},V_2=\{v_2,v_4,...,v_n\}\cup\{v_{n+3},v_{n+5},...,v_{2n+1}\}\cup\{v_{2n+2},v_{2n+3},...,v_{3n+1}\},V_3=\{v_3,v_5,...,v_{n+1}\}\cup\{v_{n+2},v_{n+4},...,v_{2n}\}$. Now if n is odd, let $C=\{V_1,V_2,V_3,V_4\}$ be the coloring of Sf_n in which $V_1=\{v_1\},V_2=\{v_2,v_4,...,v_{n-1}\}\cup\{v_{n+3},v_{n+5},...,v_{2n}\}\cup\{v_{2n+2},v_{2n+3},...,v_{3n+1}\},V_3=\{v_3,v_5,...,v_n\}\cup\{v_{n+2},v_{n+4},...,v_{2n+1}\}$ and $V_4=\{v_{n+1}\}$,then each vertex $v_i(2\le i\le 3n+1)$ is adjacent to every vertex of color class V_1 and vertex v_1 is adjacent to all vertices of color classes V_2 and V_3 (when n is even) or is adjacent to all vertices of color classes V_2 , V_3 and V_4 (when n is odd). Therefore the coloring C is a χ_d^t -coloring of (Sf_n) and hence $\chi_d^t((Sf_n))\le 3$ (when n is even) . $\chi_d^t(Sf_n)\le 4$ (when n is odd). Thus

$$\chi_d^t(Sf_n)=\begin{cases}3 & if\ \ n\ is\ even\\ 4 & if\ \ n\ is\ odd.\end{cases}$$

(vii) Let B_n be the barbell graph with $V(B_n)=\{v_1,v_2,...,v_{2n}\}$ where $\{v_1,v_2,...,v_{2n}\}$ be the vertices in first copy of K_n and $\{v_{n+1},v_{n+2},...,v_{2n}\}$ be the vertices of second copy. Let v_1v_{n+1} be the bridge of B_n. By Theorem ۱/۵ (i) we have $\chi_d(B_n)=n+1$. Now it follows from Theorem ۱/۱ that $\chi_d^t(B_n)\ge n+1$. Hence it is sufficient to show that $\chi_d^t(B_n)\le n+1$. Define a proper coloring C of B_n in which $V_1=\{v_1\},V_{i+1}=\{v_i,v_{n+i}:2\le i\le n\}$ and $V_2=\{v_{n+1}\}$. Since $v_i(2\le i\le n+1)$ is adjacent to all vertices of color class V_1 and $v_1,v_j(n+2\le j\le 2n)$ are adjacent to all vertices of color class V_2, then the coloring C is a χ_d^t -coloring of B_n and $\chi_d^t(B_n) \le n+1$. Hence $\chi_d^t(B_n) = n+1$.

(viii) For $n=2$ we have two copies of K_2 and for each copy of K_2 require two different colors. Then $\chi_d^t(S_n^0) = 4$ for $n=2$. Now for $n \ge 3$, let $V(S_n^0) = \{v_1, v_2, ..., v_n, u_1, u_2, ..u_n\}$. By Theorem ۱/۵ (ii) we have $\chi_d(S_n^0) = 4$. Now it follows from Theorem ۱/۱ that $\chi_d^t(S_n^0) \ge 4$. Hence it is sufficient to show that $\chi_d^t(S_n^0) \le 4$. Let $C = \{V_1, V_2, V_3, V_4\}$ be the coloring of S_n^0 in which $V_1 = \{v_1, v_2, ..., v_{n-1}\}, V_2 = \{v_n\}, V_3 = \{u_1, u_2, ..., u_{n-1}\}$ and $V_4 = \{u_n\}$. Then the vertex $v_i (1 \le i \le n-1)$ is adjacent to every vertex of a color class V_4,the vertex v_n is adjacent to every vertex of a color class V_3, each vertex in the set $\{u_1, u_2, ..., u_{n-1}\}$ is adjacent to every vertex of a color class V_2 and the vertex u_n is adjacent to every vertex of a color class V_1. So the coloring C is a χ_d^t -coloring of S_n^0 and $\chi_d^t(S_n^0) \le 4$. Hence $\chi_d^t(S_n^0) = 4$.

(ix) Consider the windmill graph $G = W_n^{(m)}$ formed by m-copies of the complete graph K_n with $V(W_n^{(m)}) = \bigcup_{i=1}^m \{v_1^i, v_2^i, ..., v_n^i\}$ where for each $i \in \{1,2,...,n\}$, $\{v_1^i, v_2^i, ..., v_n^i\}$ be the vertices in i-th copy of K_n and $v_1^1 = v_1^2 = ... = v_1^m$ is a common vertex. By Theorem ۱/۶, we have $\chi_d(W_n^{(m)}) = n$. Now it follows from Theorem ۱/۱ that $\chi_d^t(W_n^{(m)}) \ge n$. Hence it is sufficient to show that $\chi_d^t(W_n^{(m)}) \le n$. Let $C = \{V_1, V_2, ..., V_n\}$ be the coloring of $W_n^{(m)}$ in which $V_1 = \{v_1^i\} (1 \le i \le m)$ and for each $2 \le i \le n$, $V_i = \bigcup_{j=1}^m \{v_i^j\}$. Since each vertex in the set $\bigcup_{i=1}^m \{v_2^i, ..., v_n^i\}$ is adjacent to all vertices of a color class V_1 and a common vertex $\{v_1^i\}$ for $1 \le i \le m$, is adjacent to every vertex of a color class V_j, $2 \le j \le n$, then the coloring C is a χ_d^t -coloring of $W_n^{(m)}$ and $\chi_d^t(W_n^{(m)}) \le n$. Hence $\chi_d^t(W_n^{(m)}) = n$.

(x) Let $V(\overline{K_2(a_1, a_2)}) = \{v_i : 1 \le i \le a_1 + a_2 + 2\}$ where $\{v_1, v_2\}$ be the vertices of $\overline{K_2}$ and $\{v_3, v_4, ...v_{2+a_1}\}$ are adjacent to the vertex v_2 and $\{v_{a_1+3}, v_{a_1+4}, ..., v_{a_1+a_2+2}\}$ are adjacent to the vertex v_1.

Let

$$E(\overline{K_2(a_1, a_2)}) = \{v_1 v_i : a_1 + 3 \le i \le a_1 + a_2 + 2\} \cup \{v_2 v_j : 3 \le j \le a_1 + 2\} \cup \{v_i v_j : 3 \le i, j \le a_1 + a_2 + 2, i \ne j\}$$

.

If $a_1 = a_2 = 1$, then $\overline{K_2(1,1)}$ is a path of order 4 and theorem 2.1(iii)implies $\chi_d^t(\overline{K_2(1,1)}) = 3$.

If $a_1 = 2$ and $a_2 = 1$, consider the coloring C of $\overline{K_2(2,1)}$ in which $V_1 = \{v_1, v_3\}, V_2 = \{v_4\}, V_3 = \{v_2\}$ and $V_4 = \{v_5\}$. Then this coloring is a is a χ_d^t - coloring of $\overline{K_2(2,1)}$ and hence $\chi_d^t(\overline{K_2(2,1)}) = 4$.

Now for $a_1, a_2 \ge 2$, let $C = \{V_1, V_2, ..., V_{a_1+a_2}\}$ be the class colorings of $\overline{K_2(a_1, a_2)}$ in which $V_1=\{v_1,v_3\}, V_i=\{v_{i+2}\}(2\le i\le a_1+a_2-1)$ and $V_{a_1+a_2}=\{v_2,v_{a_1+a_2+2}\}$. Since v_i $(2\le i\le a_1+a_2+2, i\ne 4)$ is adjacent to all vertices of color class V_2 and the vertices v_1,v_4 are adjacent to every vertex of color class $V_i(a_1+1\le i\le a_1+a_2-1)$, then $\chi_d^t(\overline{K_2(a_1,a_2)})$ is a total dominator coloring and hence $\chi_d^t(\overline{K_2(a_1,a_2)})\le a_1+a_2$. Also since $\{v_3,v_4,\ldots,v_{a_1+a_2+2}\}$ is the complete graph of order a_1+a_2, it follows that at least a_1+a_2 different colors are needed to color this vertices. Thus $\chi_d^t(\overline{K_2(a_1,a_2)})\ge a_1+a_2$. Therefore $\chi_d^t(\overline{K_2(a_1,a_2)})=a_1+a_2$ for $a_1,a_2\ge 2$.

(xi) Let $V(R_2)=\bigcup_{i=1}^{2}\{u_1^i,u_2^i,\ldots,u_m^i,v_1^i,v_2^i,\ldots,v_n^i\}$, where $\{u_1^i,u_2^i,\ldots,u_m^i\}$ be the one partite set in the $i-$th copy and $\{v_1^i,v_2^i,\ldots,v_n^i\}$ be the other partite set in the $i-$th copy. Let $u_m^1v_1^2$ be the bridge of R_2. Let $S=\{u_m^1,v_n^1,u_1^2,v_1^2\}$. Since each vertex in $V(R_2)$ is adjacent to at least one vertex in S, the set S is a total dominating set. Also the set S is a γ_t-set, since at least one vertex of each partite set is needed for any total dominating set. Hence $\gamma_t(R_2)=4$ and it follows from Theorem 1.1, $\chi_d^t(R_2)\ge 4$. Consider a proper coloring C of R_2 in which $V_1=\{u_1^1,u_2^1,\ldots,u_m^1\}, V_2=\{v_1^1,v_2^1,\ldots,v_n^1\}, V_3=\{u_1^2,u_2^2,\ldots,u_m^2\}$ and $V_4=\{v_1^2,v_2^2,\ldots,v_n^2\}$. Then the vertices $\{u_1^i,u_2^i,\ldots,u_m^i\}$ is adjacent to every vertex of a color class $V_{2i}(1\le i\le 2)$, each vertex in the set $\{v_1^1,v_2^1,\ldots,v_n^1\}$ is adjacent to every vertex of a color class V_1 and the set $\{v_1^2,v_2^2,\ldots,v_n^2\}$ is adjacent to every vertex of a color class V_3. Hence the coloring C is a χ_d^t-coloring of R_2 and $\chi_d^t(R_2)\le 4$. Therefore $\chi_d^t(R_2)=4$.

3 Further research

To finish our discussion we state some open problems for further research..

Probleme ١. Characterize graphs G for which $\chi_d^t(G)=\chi_d(G)$ or $\chi_d^t(G)=\gamma_t(G)$.

Probleme ٢. Characterize graphs G for which $\chi_d^t(G)=\chi_d(G)=\gamma_t(G)$.

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