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Bounds on double Roman domination number of graphs

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Let $G=(V,E)$ be a simple graph. A double Roman dominating function of a graph G is a function $f:V\to\{0,1,2,3\}$ having the property that if $f(v)=0$, then the vertex v must have at least two neighbors w_1,w_2 such that $f(w_1)=f(w_2)=2$ or one neighbor w such that $f(w)=3$; and if $f(v)=1$, then the vertex v must have at least one neighbor w such that $f(w)\geq 2$. The weight of a double Roman dominating function is the sum $w_f=\sum_{v\in V(G)} f(v)$, and the minimum weight of w_f for every double Roman dominating function f on G is called double Roman domination number of G. We denote this number with $\gamma_{dR}(G)$. In this paper; we obtain some new lower and upper bounds of double Roman domination number of graphs.

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1 Introduction

Graph domination applies naturally too many tasks, including facility location and network construction, for example, in constructing a cellular phone network, one needs to

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choose locations for the towers to cover a large region as cheaply as possible. Many variants of domination have been studied extensively and have applications such as constructions of error-correcting codes for digital communication and efficient data routing in wireless networks. The original study of Roman domination was motivated by the defense strategies used to defend the Roman Empire during the reign of Emperor Constantine the Great, 274-337 A.D. He decreed that for all cities in the Roman Empire, at most two legions should be stationed. Further, if a location having no legions was attacked, then it must be within the vicinity of at least one city at which two legions were stationed, so that one of the two legions could be sent to defend the attacked city. This part of history of the Roman Empire gave rise to the mathematical concept of Roman domination, as originally defined and discussed by Stewart [7] in 1999, and ReVelle and Rosing [6] in 2000.

Let $G=(V,E)$ be a graph of order $n=|V(G)|$. We consider the following game. You are allowed to buy as many tokens from a bank as you like, at a cost of 1 dollar each. For example, Suppose you buy k tokens. You then place the tokens on some subset of k vertices of V. For each vertex of G which has no token on it, But is adjacent to a vertex with a token on it, you received 1 dollar from the bank. Your objective is to maximize your profit, that is, the total value received from the bank minus the cost of the tokens bought. Let $bd(X)$ be the set of vertices in $V-X$ that have a neighbour in the set X. For a nonempty subset $X \subseteq V$ we write $C(X)=V-(X \cup bd(X))$. Based on this game, We define the differential of a set X to be $\partial(X)=|bd(X)|-|X|$ [5], and the differential of a graph to be equal to $\partial(G)=max\{\partial(X): X \subseteq V\}$. One of variations of a differential of graphs, is B-differential of graphs. We denote this parameter of graph with $\Psi(G)$ and we define $\Psi(G)=max\{|bd(X)|: X \subseteq V\}$ [5]. A graph G is said to be dominant differential [3] if it contains a ∂-set which is also a dominating set. Some examples of dominant differential graphs are complete graphs, star graphs, wheel graphs, path graphs P_n and cycle graphs C_n with $n=3k$ or $n=3k+2$. A graph G is said to be double Roman graph if $\gamma_{dr}(G)=3\gamma(G)bhh$. We denote minimum degree of vertices of graph G with $\delta(G)$ and maximum degree of vertices of graph G with $\Delta(G)$. The open neighborhood of a vertex $v \in V(G)$ is the set $N(v)=\{u: uv \in E(G)\}$. The open neighborhood of a set $S \subseteq V$ is the set $N(S)=\bigcup\{N(v): v \in S\}$. The closed neighborhood of a set $S \subseteq V$ is the set $N[S]=N(S)\bigcup S$. Let E_v be the set of edges incident with v in G that is, $E_v=\{uv \in E(G): u \in N_G(v)\}$. We denote the degree of v by $d_G(v)=|E_v|$. A vertex of degree zero is called an isolated vertex. Given a set $S \subseteq V$ the private neighborhood $pn[v,S]$ of $v \in S$ is defined by $pn[v,S]=N[v]-N[S-\{v\}]$, equivalently, $pn[v,S]=\{u \in V: N[u] \cap S=\{v\}\}$. Each vertex in $pn[v,S]$ is called a private neighbor of v.

The external private neighborhood $epn(v,S)$ of v with respect to S consists of those private neighbors of v in $V-S$. Thus $epn(v,S)=pn[v,S]\cap(V-S)\, l.$

A set $S \subseteq V$ is a dominating set if $N[S]=V$. A domination number $\gamma(G)$ is the minimum cardinality a $\gamma(G)$-Set. A graph G has property EPN if for every $\gamma(G)$-set S and for every $v \in S$, $epn(v,S) \neq \varnothing$. We call a tree with property EPN, an EPN-tree [5].

For a graph $G=(V,E)$ let $V_i=\{v\in V(G): f(v)=i\}$. A Roman dominating function on graph G is a function $f:V\to\{0,1,2\}$ such that if $v\in V_0$ for some $v\in V$, then there exists $w\in N(v)$ such that $w\in V_2$. The weight of a Roman dominating function is the sum $w_f=\sum_{v\in V(G)} f(v)$, and the minimum weight of w_f for every Roman dominating function f on G is called Roman domination number of G. We denote this number with $\gamma_R(G)$. We say that graph G is Roman graph if We have $\gamma_R(G)=2\gamma(G)$.

A double Roman dominating function on graph G is a function $f:V\to\{0,1,2,3\}$ such that the following conditions are met:

(a) if $f(v)=0$, then vertex v must have at least two neighbors in V_2 or one neighbor in V_3.

(b) if $f(v)=1$, then vertex v must have at least one neighbor in $V_2\bigcup V_3$.

The weight of a double Roman dominating function is the sum $w_f=\sum_{v\in V(G)} f(v)$, and the minimum weight of w_f for every double Roman dominating function f on G is called double Roman domination number of G. We denote this number with $\gamma_{dr}(G)$. Robert A. Beeler et al., in [1] have been studied the double Roman graph.

2 Known results.

The following results are important for our investigations.

Theorem A [5]. For any graph G of order n, $\Psi(G)=n-\gamma(G)$.

Theorem B [3]. A graph is dominant differential if and only if $\partial(G)=n-2\gamma(G)$.

Theorem C [2]. If G is a graph of order n, then $\gamma_R(G)=n-\partial(G)$.

Theorem D [2]. If G is a graph of order $n\ge 3$, then

$$n-\gamma\ (G)(\Delta(G)-1)\le \gamma_R(G)\le n-\frac{\gamma(G)}{2}.$$

3 Upper and lower bounds

Theorem 1 *Let G be a simple connected graph of order n. If γ_{dR}-function f on G exists such that $f=(V_0,V_1,V_2,V_3)$ and $V_3\ne\varnothing$, then $\gamma_{dR}(G)\ge 2n+1-2\Psi(G)$.*

Proof. For every double Roman domination function $g=(W_0,W_1,W_2,W_3)$ on graph G we consider γ_{dR} -function $h=(U_0,U_1,U_2,U_3)$ on G such that $U_0=bd(W_0), U_1=\varnothing, U_2=C(W_3), U_3=W_3$. Clearly we have $g(W)\ge h(U)$. Therefore,

$$\gamma_{dR}(G)=min\{h(U):U=(U_0,U_1,U_2,U_3)\}=min\{h(U):U=(U_0,U_1,U_2,U_3),U_3\ne\varnothing\}=$$

$$= min\{2|U_2|+3|U_3|: U_2, U_3 \subseteq V\} = min\{2|C(W_3)|+3|W_3|: W_3 \subseteq V, W_3 \neq \varnothing\}$$

$$= min\{2|C(W_3)|+3|W_3|+2|bd(W_3)|-2|bd(W_3)|: W_3 \subseteq V\} =$$

$$= min\{2|C(W_3)|+2|W_3|+2|bd(W_3)|+|W_3|-2|bd(W_3)|: W_3 \subseteq V\} = min\{2n+|W_3|-2|bd(W_3)|: W_3 \subseteq V\}$$

$$= 2n - max\{2|bd(W_3)|-|W_3|: W_3 \subseteq V\}$$

Now since $|W_3| \geq 1$, we have,

$$\gamma_{dR}(G) \geq 2n - max\{2|bd(W_3)|-1: W_3 \subseteq V\}$$

$$= 2n+1-2max\{|bd(W_3)|: W_3 \subseteq V\} = 2n+1-2\Psi(G).$$

Theorem 2 *Let G be a simple connected graph of order n. if γ_{dR}-function f on G exists such that $f = (V_0, V_1, V_2, V_3)$ and $V_3 \neq \phi$, then $\gamma_{dR}(G) \geq 2\gamma(G)+1$.*

Proof. By Theorem 1 and Theorem A, we have: $\gamma_{dR}(G) \geq 2\gamma(G)+1$.

Theorem 3 *Let T be a nontrivial tree of order n. Then*

$$\gamma_{dR}(T) \geq 2\gamma(T)+1$$

.

Proof. Suppose on the contrary that for every γ_{dR}-function $f = (V_0, V_1, V_2, V_3)$ on T, we have $V_3 = \varnothing$. Therefore $V_T = V_0 \bigcup V_2$. Now we consider two cases as follows:

Case1. If at least two vertices v_1, v_2 in set V_0 are adjacent, then by definition of double Roman domination function we must have distinct vertices v_3, v_4, v_5, v_6 in set of V_2 such that v_1 is adjacent with v_3, v_4 and v_2 is adjacent with v_5, v_6. Since, if the vertices were not distinct, then there was exist a cycle. Now we assign label 3 to vertices v_1, v_2 and label 0 to vertices v_3, v_4, v_5, v_6. Therefore we achieve a double Roman domination function with less weight.

Case2. If every two vertices v_0, v_1 in V_0 are not adjacent, then there is not exist any cycle in a tree, therefore by definition of double Roman domination function we must have two vertices v_0, v_1 with at least three vertices of label 2 are adjacent. Now we assign label 3 to vertices v_0, v_1 and assign label 0 to three vertices of label 2 which are adjacent. Therefore we achieve a double Roman domination function at most with equal weight. Hence according to upper two cases we achieve contradiction. Thus according to Theorem 2 we have,

$$\gamma_{dR}(T) \geq 2\gamma(T)+1.$$

Theorem 4 *If a graph G is dominant differential and without isolated vertices, then*

$$\gamma_{dR}(G) \geq 2n-\Psi(G)-\partial(G).$$

Proof. By proof of Theorem 1 we have,

$$\gamma_{dR}(G) = min\{h(U): U=(U_0,U_1,U_2,U_3)\} =$$

$$= 2n - max\{2|bd(W_3)|-|W_3|: W_3 \subseteq V\}$$

$$= 2n - max\{\Psi(W_3)+\partial(W_3): W_3 \subseteq V\}$$

But since a graph G is dominant differential and without isolated vertices thus we have,

$$\gamma(G) \leq n/2, \partial(G) = n-2\gamma(G).$$

Therefore we have $\gamma(G) \geq 0$. Thus,

$$\gamma_{dR}(G) \geq 2n - max\{\Psi(G)+\partial(G)\} = 2n-\Psi(G)+\partial(G).$$

Theorem 5 *If a graph G is dominated differential and without isolated vertices, then G is a double Roman graph.*

Proof. By Theorems A , B and 4 we have,

$$\gamma_{dR}(G) \geq 2n-\Psi(G)+\partial(G) = 2n-(n-\gamma(G))-(n-2\gamma(G))$$

Thus,

$$\gamma_{dR}(G) \geq 3\gamma(G)$$

But for every graph G we have $\gamma_{dR}(G) \leq 3\gamma(G)$ therefore $\gamma_{dR}(G) = 3\gamma(G)$.

Theorem 6 *If a tree T is a double Roman, then it is EPN-tree.*

Proof. We prove, if a tree T is double Roman, then T is EPN-tree. We suppose that T is not EPN-tree. Thus we have for some $\gamma(T)$-set S and for some vertex $v_0 \in S$, $epn(v_0,S) = \emptyset$. We can suppose that $S=\{v_0,v_1,...,v_{\gamma(T)-1}\}$ and $|S| \geq 2$. Therefore $pn[v_0,S] \bigcap (V-S) = \emptyset$. Thus $pn[v_0,S] \subseteq S$. Now clearly, we can say that a dominating set S is a minimal dominating set if and only if every vertex in S has at least one private neighbor. Since S is a $\gamma(T)$-set, then for every $i \in \{0,1,...,\gamma(T)-1\}$ we have $pn[v_i,S] \neq \emptyset$. Thus there exists a vertex w such that $N[w] \bigcap S = \{v_0\}$. But we have $pn[v_0,S] \subseteq S$, hence $w \in S$. Now if we have $w \neq v_0$, then we should have $w = v_i$ for some $i \neq 0$. On the other hand, every neighbor of vertex v_0 in set of $V-S$ is adjacent to at least one another vertex in set of S because the vertex of v_0 does not have private neighbor in set of $V-S$. Now we can delete the vertex of v_0 from the set of S such that this new set will be a $\gamma(T)$-set with smaller cardinality. Therefore we have $w = v_0$. Hence v_0 is an isolated vertex in set of S. But T is a tree thus T is connected. Therefore there exists a vertex $w \in (V - S)$ such that $w \in N(v_0)$. But we have $pn[v_0, S] \subseteq S$ thus we must have a vertex $v_1 \in S$ such that $w \in N(v_1)$. Now if there exists another vertex $w_1 \in (V - S)$ such that $w_1 \in N[v_0]$, then we should have a vertex for example $v_2 \in S$ such that $w_1 \in N[v_2]$ because a vertex v_0 does not have any private neighbour in set of $V - S$. Now we can define function $f : V \to \{0,1,2,3\}$ such that $f(v_0) = 2, f(v_1) = 3, \ldots, f(v_{\gamma(T)-1}) = 3$ and $f(w) = 0$ for every $w \in (V - S)$. Clearly f is a double Roman domination function on T such that $w_f < 3\gamma(T)$. This inequality contradicts with being double Roman tree.

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