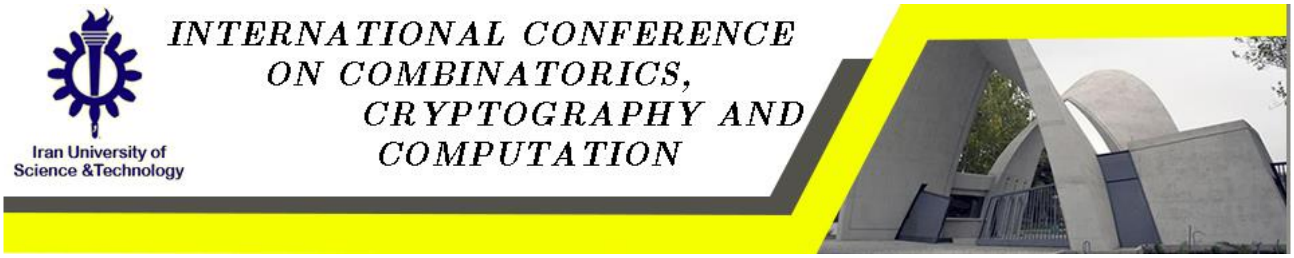

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A fast numerical algorithm based on alternative orthogonal polynomials for fractional optimal control problems

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ABSTRACT

In this paper, we focus on alternative Legendre polynomials (ALPs) in fractional calculus area and obtain the operational matrix of the Riemann-Liouville fractional integration for the first time. To solve the problem, first the fractional optimal control problem (FOCP) is transformed into an equivalent variational problem, then using the alternative Legendre polynomials basis, the problem is reduced to the problem of solving a system of algebraic equations. With the aid of an operational matrix of Riemann-Liouville fractional integration, Gauss quadrature formula and Newton's iterative method for solving a system of algebraic equations, the problem is solved approximately. Some examples are given to demonstrate the validity and applicability of the our technique and a comparison is made with the existing results.

KEYWORDS: Fractional optimal control problem, Alternative Legendre polynomials, Caputo fractional derivative, Numerical method, Operational matrix.

1 INTRODUCTION

Fractional order dynamics appear in several problems in science and engineering such as visco-elastic materials [1], economics [2], continuum and statistical mechanics [3], solid mechanics [4], bioengineering [5], dynamics of interfaces between nanoparticles and substrates [6], etc. Interested reader in fractional calculus and its applications can see [7, 8].

The optimal control problems (OCPs) have a long history and they have naturally occur in engineering, science, geometry, pure and applied mathematics examples where extremization of functional, such as, Lagrangian, strain, potential and total energy and etc. The general definition of an optimal control prob- lem requires the extremizing of a performance criterion or a cost function over an admissible set of control and state functions. The system is subject to constrained dynamics and state and control variables. A fractional optimal control problem (FOCP) is an optimal control problem in which the performance index or the differential equation governing the dynamic of the system or both contain at least one fractional derivative operator. A general formulation and a solution scheme for FOCPs were first introduced in [9], where fractional derivatives were introduced in the Riemann-Liouville sense, and FOCP formulation was expressed using the fractional variational principle and the Lagrange multiplier technique. The author of [10] considered a one dimensional non-feedback system to solve FOCP by using eigenfunctions method. In [11] the fractional variational problems were solved using a numerical technique. Rahimkhani et al. [12] used the Mn̈tz-Legendre orthonormal basis and the properties of Rayleigh-Ritz method for fractional calculus to reduce FVPs to solve a system of algebraic equations which solved using Newton's iterative method. Lotfi et al. [13] presented a numerical direct method for solving a general class of FOCPs. The paper [14] discussed on FCOPs with Riemann-Liouville fractional derivative and their solutions by means of rational approximation. Agrawal [15] achieved the necessary conditions of optimization for FOCPs with the Caputo fractional derivative. In [16], a new numerical method based on Bernoulli wavelets for solving the fractional optimal control problems (DFOCPs) with quadratic performance index presented. Rahimkhani et al. [18] presented a new method to solve a class of two-dimensional fractional optimal control problems. The authors of [17] solved two-dimensional FOCPs in polar coordinates by separation variables method. Fractional diffusion problems in three-dimensional and in the spherical coordinates are discussed precisely in [19, 20].

Approximation by orthogonal family of functions have found wide application in science and engineering. The most frequently used orthogonal functions are sine-cosine functions, block-pulse functions, Legendre functions, Laguerre and Chebyshev functions, wavelets [21, 22]. Although, few papers reported application of alternative Legendre polynomials to solve differential equations. The alternative Legendre polynomials are a family of generalized orthogonal polynomials, which these polynomials were introduced and investigated in [23]. In [24] the operational matrices of integration and the product for the alternative Legendre polynomials are derived and used for solving the nonlinear Volterra-Fredholm -Hammerstein integral equations. In [25] Sequences of orthogonal polynomials that are alternative to the Jacobi polynomials on the interval [0, 1] are defined and their properties are established.

In the present paper, we introduce a computational method based on alternative Legendre polynomials to solve the fractional optimal control problems. First we transform the FOCP into an equivalent variational problem, then we expand the unknown function with alternative Legendre polynomials basis and the unknown coefficients. Finally, the operational matrix of fractional integration is utilized to achieve a system of algebraic equations in the unknown coefficients. These coefficients are determined in such a way that the necessary conditions for extremization are imposed. Also, illustrative examples are included to demonstrate the applicability of the novel strategy.

The present article is organized as follows. In section 2, we introduce some basic definitions of fractional calculus and alternative Legendre polynomials. In section 3, the alternative Legendre polynomials operational matrix of fractional integration is given. Section 4, is devoted to the problem statement and the numerical method for solving FOCPs. In section 5, we report our numerical results and demonstrate the accuracy of the proposed method by considering numerical examples. A conclusion is given in section 6.

2 PRELIMINARIES AND NOTATIONS

In this section, we will recall some basic definitions and auxiliary results that will be needed later in our discussion. Here we give the standard definitions of Riemann-Liouville fractional integrals and Caputo fractional derivatives.

2.1 The fractional integral and derivative

Deftnition 1. *A real function f(t), $t > 0$, is said to be in the space C_μ, $\mu \in R$ if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$. Clearly, $C_\mu \subset C_\beta$ if $\beta < \mu$ [26].*

Deftnition 2. *A function f(t), $t > 0$, is said to be in the space C^n if and only if $f^{(n)} \in C_\mu$, $n \in N$ [26].*

Deftnition 3. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function, $f \in C_\mu$, $\mu \geq -1$, is defined as [26]*

$$I^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, & \alpha > 0, t > 0, \\ f(t), & \alpha = 0 \end{cases} \quad (1)$$

For the Riemann-Liouville fractional integral we have [26]

1. $I^\alpha (\lambda_1 f(t) + \lambda_2 g(t)) = \lambda_1 I^\alpha f(t) + \lambda_2 I^\alpha g(t),$
2. $I^\alpha t^\beta = \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\alpha+\beta}, \beta > -1,$

where λ_1, λ_2 and c are constants.

Deftnition 4. *Caputo's fractional derivative of order α is defined as [26]*

$$D^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} ds, \qquad n-1 < \alpha \leq n, n \in \mathrm{N}. \quad (2)$$

For the Caputo derivative we have [26]

1. $D^\alpha I^\alpha f(t) = f(t),$
2. $I^\alpha D^\alpha f(t) = f(t) - \sum_{i=0}^{n-1} f^{(i)}(0) \dfrac{t^i}{i!},$
3. $D^\alpha \lambda = 0,$

where λ is constant.

2.2 Alternative Legendre polynomials

Let m be a fixed non-negative integer. The set $P_m = \{p_{mi}(t)\}_{i=0}^m$ of alternative Legendre polynomials were recently introduced by [23] as:

$$p_{mi}(t)=\sum_{r=0}^{m-i}(-1)^r\binom{m-i}{r}\binom{m+i+r+1}{m-i}t^{i+r}$$

$$=\sum_{r=i}^{m}(-1)^{r-i}\binom{m-i}{r-i}\binom{m+r+1}{m-i}t^r, \qquad i=0,1,\ldots,m. \qquad (3)$$

These polynomials are orthogonal over the interval [0, 1] with respect to the weight function $\omega(\mathrm{t})=1$, and satisfy the orthogonality relationships [24]

$$\int_0^1 p_{mk}(t)p_{ml}(t)dt=\frac{1}{k+l+1}\delta_{kl}, \qquad k,l=0,1,\ldots,m. \qquad (4)$$

Here δ_{kl} denotes the Kronecker delta. It should be noted that, in contrast to common sets of orthogonal polynomials, every member in P_m has degree m. For example, when m = 3 we have:

$$p_{30}(t)=4-30t+60t^2-35t^3,$$
$$p_{31}(t)=10t-30t^2+21t^3,$$
$$p_{32}(t)=6t^2-7t^3,$$
$$p_{33}(t)=t^3. \qquad (5)$$

2.3 Function approximation

Suppose $f \in L^2[0, 1]$ can be expanded in terms of the alternative Legendre polynomials as

$$f(t)=\sum_{i=0}^{\infty}c_i p_{mi}(t), \qquad (6)$$

where the coefficients c_i are given by

$$c_i=\langle f, p_{mi}\rangle=(2i+1)\int_0^1 f(t)p_{mi}(t)dt,$$

where $<, >$ denotes the inner product in $L^2[0, 1]$. If the infinite series in Eq. (6) is truncated, then it can be written as

$$f(t)\simeq\sum_{i=0}^{m}c_i p_{mi}(t)=C^T\Phi(t), \qquad (7)$$

where T indicates transposition, C and $\Phi(t)$ are $(m + 1) \times 1$ vectors given by

$$C=[c_0,c_1,c_2,\ldots,c_m]^T,$$

and

$$\Phi(t)=[p_{m0}(t),p_{m1}(t),p_{m2}(t)\ldots,p_{mm}(t)]^T.$$

2.4 The alternative Legendre polynomials operational matrix of the fractional integration

3 THE ALTERNATIVE LEGENDRE POLYNOMIALS OPERATIONAL MATRIX OF THE FRACTIONAL INTEGRATION

In this section, we derive $(m+1)\times(m+1)$ the Riemann-Liouville fractional operational matrix of integration $F^{(\alpha)}$ expressed by:

$$I^{\alpha}\Phi(t)=F^{(\alpha)}\Phi(t), \qquad (8)$$

By using Eq. (3) and linearity of Riemann-Liouville fractional integral, for i = 0, 1, . . . , m we have:

$$I^{\alpha}p_{mi}(t)=\sum_{r=i}^{m}(-1)^{r-i}\begin{pmatrix} m-i \\ r-i \end{pmatrix}\begin{pmatrix} m+r+1 \\ m-i \end{pmatrix}I^{\alpha}(t^{r})$$

$$=\sum_{r=i}^{m}(-1)^{r-i}\begin{pmatrix} m-i \\ r-i \end{pmatrix}\begin{pmatrix} m+r+1 \\ m-i \end{pmatrix}\frac{\Gamma(r+1)}{\Gamma(r+\alpha+1)}t^{r+\alpha}$$

$$=\sum_{r=i}^{m}b_{mi,r}^{(\alpha)}t^{\alpha+r}, \qquad (9)$$

where

$$b_{mi,r}^{(\alpha)}=(-1)^{r-i}\begin{pmatrix} m-i \\ r-i \end{pmatrix}\begin{pmatrix} m+r+1 \\ m-i \end{pmatrix}\frac{\Gamma(r+1)}{\Gamma(r+\alpha+1)}.$$

Now, approximating $t^{r+\alpha}$ by m + 1 term of alternative Legendre polynomials, we get

$$t^{r+\alpha}\simeq\sum_{j=0}^{m}c_{r,j}^{(\alpha)}p_{mj}(t). \qquad (10)$$

Substituting Eq. (10) into Eq. (9) for i = 0, 1, ... , m, we obtain

$$I^{\alpha}p_{mi}(t)\simeq\sum_{r=i}^{m}b_{mi,r}^{(\alpha)}\sum_{j=0}^{m}c_{r,j}^{(\alpha)}p_{mj}(t)=\sum_{j=0}^{m}\Big(\sum_{r=i}^{m}\omega_{mi,j,r}^{(\alpha)}\Big)p_{mj}(t), \qquad (11)$$

where

$$\omega_{mi,j,r}^{(\alpha)}=b_{mi,r}^{(\alpha)}c_{r,j}^{(\alpha)}.$$

Eq. (11) can be rewritten as

$$I^{\alpha}p_{mi}(t)\simeq[\sum_{r=i}^{m}\omega_{mi,0,r}^{(\alpha)},\sum_{r=i}^{m}\omega_{mi,1,r}^{(\alpha)},\ldots,\sum_{r=i}^{m}\omega_{mi,m,r}^{(\alpha)}]\Phi(t), \qquad i=0,1,\ldots,m.$$

Therefore, we get

$$F^{(\alpha)} = \begin{bmatrix} \sum_{r=0}^{m} \omega^{(\alpha)}_{m0,0,r} & \sum_{r=0}^{m} \omega^{(\alpha)}_{m0,1,r} & \dots & \cdots & \sum_{r=0}^{m} \omega^{(\alpha)}_{m0,m,r} \\ \sum_{r=1}^{m} \omega^{(\alpha)}_{m1,0,r} & \sum_{r=1}^{m} \omega^{(\alpha)}_{m1,1,r} & \dots & \cdots & \sum_{r=1}^{m} \omega^{(\alpha)}_{m1,m,r} \\ \vdots & \vdots & & \cdots & \vdots \\ \sum_{r=m-1}^{m} \omega^{(\alpha)}_{m(m-1),0,r} & \sum_{r=m-1}^{m} \omega^{(\alpha)}_{m(m-1),1,r} & \dots & \cdots & \sum_{r=m-1}^{m} \omega^{(\alpha)}_{m(m-1),m,r} \\ \omega^{(\alpha)}_{mm,0,m} & \omega^{(\alpha)}_{mm,1,m} & & \cdots & \omega^{(\alpha)}_{mm,m,m} \end{bmatrix}$$

4 PROBLEM STATEMENT AND APPROXIMATION METHOD

In the current section, we use the operational matrix of the fractional integration of alternative Legendre polynomials, described in the Riemann-Liouville sense, in combination with the Gauss quadrature formula for minimizing the quadratic cost function

$$min(max) \qquad J[u] = \int_0^1 f(t, x(t), u(t))dt, \qquad (12)$$

subject to

$$D^{\alpha} x(t) = g(t, x(t)) + b(t)u(t), \qquad n-1 \le \alpha \le n, n = 2,3,\dots \qquad (13)$$
$$x(0) = x_0, x'(0) = x_1, \dots, x^{(\lceil \alpha \rceil - 1)}(0) = x_{\lceil \alpha \rceil - 1}.$$

Here f, g and $b \neq 0$ are smooth functions of their arguments. Computing the control function from relation (13) and substituting into the performance index J yields

$$min(max) J[x] = \int_0^1 f(t, x(t), \frac{1}{b(t)}(D^{\alpha} x(t) - g(t, x(t))))dt, \qquad (14)$$

$$x(0) = x_0, x'(0) = x_1, \dots, x^{(\lceil \alpha \rceil - 1)}(0) = x_{\lceil \alpha \rceil - 1}.$$

Now we solve the optimization problem (14) via alternative Legendre polynomials approximation. First, we approximate $D^{\alpha}x(t)$ and $x(t)$ as

$$D^{\alpha} x(t) \simeq C^T \Phi(t) \qquad (15)$$

and

$$x(t) \simeq C^T F^{(\alpha)} \Phi(t) + \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t^i}{i!} x_i \qquad (16)$$

By substituting the approximations (15) and (16) in the performance index (14), our aim is to optimize the functional of m variables, c_i ; $i = 0, 1, 2, \dots, m$ as

$$min(max) \qquad J[C] = \int_0^1 f(t, C^T F^{(\alpha)} \Phi(t) + \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t^i}{i!} x_i, \frac{1}{b(t)}(C^T \Phi(t) - g(t, C^T F^{(\alpha)} \Phi(t) + \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{t^i}{i!} x_i)))dt.$$

For simplicity, we let

$$min(max)J[C]=\int_0^1 \Lambda(t,C)dt, \qquad (17)$$

where

$$\Lambda(t,C)= f(t,C^T F^{(\alpha)}\Phi(t)+\sum_{i=0}^{\lceil\alpha\rceil-1}\frac{t^i}{i!}x_i, \frac{1}{b(t)}(C^T\Phi(t)-g(t,C^T F^{(\alpha)}\Phi(t)+\sum_{i=0}^{\lceil\alpha\rceil-1}\frac{t^i}{i!}x_i))).$$

The Gaussian quadrature formula is used to approximate the integration in Eq. (17). we get

$$\tilde{J}[C]=\frac{1}{2}\sum_{i=1}^{2m}\eta_i\Lambda(\frac{\tau_i+1}{2},C), \qquad (18)$$

where τ_i ; i = 1, 2, . . . , 2m; are zeros of Legendre polynomial $P_{2m}(t)$ and $\eta_i=\frac{-2}{(2m+1)P'_{2m}(\tau_i)P_{2m+1}(\tau_i)}, i=1,2,\ldots,2m.$

The necessary conditions for extremum of $\tilde{J}$ are given by

$$\frac{\partial\tilde{J}}{\partial c_i}[C]=0, \qquad i=0,1,2,\ldots,m.$$

The above relations can be solved for C using the Newton's iterative method.

5 NUMERICAL EXAMPLES

In this section, we give some numerical examples and apply the technique of section 4 for solving them. These test problems demonstrate the applicability and accuracy of our method. All the numerical computations have been done using Mathematica.

Example 1. *Consider the following FOCP as [27]*

$$\min \quad J=\frac{1}{2}\int_0^1\left[x^2(\mathrm{t})+u^2(\mathrm{t})\right]dt,$$

s.t.

$$\begin{cases} D^\alpha x(t)=tx(t)+u(t), & 0\le t\le 1, 0<\alpha\le 1, \\ x(0)=1, \end{cases}.$$

Table 1 compares the values of $\tilde{J}$, obtained via alternative orthogonal polynomials method and the results reported in [27]. The number of our basis and calculation is less than [27]. Also, Fig. 1 demonstrates the values of state and control functions for different values of α.

Table 1 The comparison of estimated value of $\tilde{J}$ for different values of α, for Example 1.

A	***Method in* [27] *for m* = 5**	***Our method for m* = 4**
0.8	0.466978	0.466979

0.9	0.475883	0.475883
0.99	0.483463	0.483463
1	0.484268	0.484268

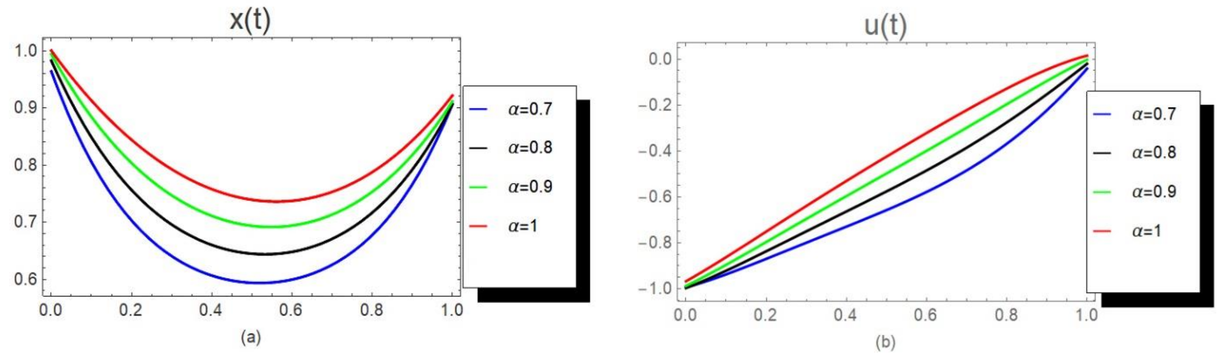

Figure 1: The comparison of $x(t)$ and $u(t)$ for m = 4, with α = 0.7, 0.8, 0.9, 1, for Example 1.

Example 2. *Consider the following FOCP as [28]*

$$\min \quad J = \int_0^1 \left[0.625x^2(\mathrm{t}) + 0.5x(\mathrm{t})\,\mathrm{u}(\mathrm{t}) + 0.5u^2(\mathrm{t})\right]dt,$$

s.t.

$$\begin{cases} D^{\alpha}x(t) = 0.5x(t) + u(t), & 0 \le t \le 1, 0 < \alpha \le 1, \\ x(0) = 1, \end{cases}.$$

that its exact optimal cost for α = 1, is J = 0.3807971, and the exact value of control variable is [28]

$$u(t) = \frac{-(\tanh(1-t) + 0.5)\cosh(1-t)}{\cosh(1)}.$$

In Table 2, we compare the value $\tilde{J}$ obtained using the proposed method with m = 3, 4 with the value of $\tilde{J}$ reported in [28] by using the Boubaker polynomials basis functions with m = 5 for different values of α. In addition, Figs. 2(a) and 2(b) demonstrate the approximation of the state variable $x(t)$ and the control variable $u(t)$ respectively, for m = 3 with different values of α.

Table 2: The comparison of estimated value of $\tilde{J}$ for different values of α, for Example 2.

***Method in* [28]*for m* = 5**		***P resent method***	
		m = 3	m = 4
0.8	0.352311	0.352307	0.352311
0.9	0.366705	0.366704	0.366705
0.99	0.379407	0.379407	0.379407
1	0.380797	0.380797	0.380797

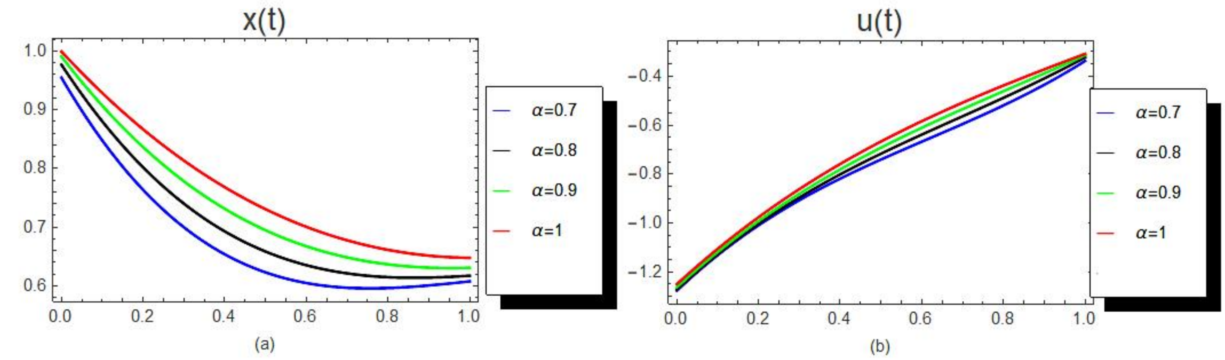

Figure 2:Approximate solution of $x(t)$ and $u(t)$ for $m = 3$, with α=0.7,0.8,0.9,1,for Example 2.

Example 3. *Consider the following FOCP as [29]*

$$\min \qquad J=\int_0^1\Big[(x(t)-t^2)^2+(u(t)+t^4-\frac{20t^{\frac{9}{10}}}{9\Gamma(\frac{9}{10})})^2\Big]dt, \qquad (19)$$

s.t.

$$\begin{cases} D^{1.1}x(t)=t^2x(t)+u(t), & 0\le t\le 1, \qquad (20)\\ x(0)=\dot{x}(0)=0, \end{cases}$$

minimize the performance index $\tilde{J}$ and the minimum $x(t)=t^2$ *and* $u(t)=\frac{20t^{\frac{9}{10}}}{9\Gamma(\frac{9}{10})}-t^4$ value is zero. Table 3, shows the values of $\tilde{J}$ obtained by the Legendre polynomials [29] together with the present method with different values of m. Also, Figs. 3(a) and 3(b) show the absolute errors of the numerical solutions of the state variable $x(t)$ and the control variable $u(t)$, respectively.

Table 3: The comparison of estimated value of $\tilde{J}$ for different values of m, for Example 3.

m	***Method in* [29]**	***Our method***
3	6.07530×10^{-6}	6.17477×10^{-6}
4	1.67255×10^{-6}	1.69123×10^{-6}
5	5.91532×10^{-7}	5.94500×10^{-7}
7	1.21966×10^{-7}	1.15859×10^{-7}
8	7.03371×10^{-8}	5.95135×10^{-8}

Example 4. *Consider the following FOCP as [29]*

$$\min \qquad J=\int_0^1\Big[(x(t)-t^{\frac{5}{2}})^4+(1+t^2)(u(t)+t^6-\frac{15\sqrt{\pi}t}{8})^2\Big]dt, \qquad (21)$$

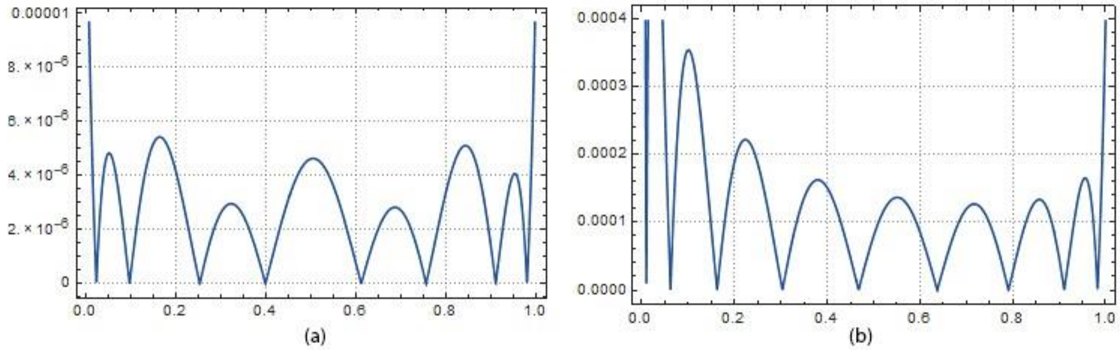

Figure 3: The absolute errors of (a): $x(t)$ and (b): $u(t)$ with m = 8 for Example 3.

$s.t.$

$$\begin{cases} D^{1.5}x(t) = tx^2(t) + u(t), & 0 \le t \le 1, \\ x(0) = \dot{x}(0) = 0. \end{cases} \tag{22}$$

For this problem we have $\bar{x}(t) = t^{\frac{5}{2}}$ as the optimal state variable and minimum value zero for the performance index $\tilde{J}$. Table 4, shows the comparison between the approximation of $\tilde{J}$ obtained by the Legendre polynomials [29] with $m = 3$ together with the present method with $m = 3, 4, 5$. Also, Fig. 4 demonstrates the absolute errors of $x(t)$ with m = 5.

Table 4: The comparison of the estimated value of $\tilde{J}$ for different values m for Example 4.

	***Method in* [29]**	***P resent method***		
		$m = 3$	$m = 4$	$m = 5$
$\tilde{J}$	2.2777×10^{-7}	2.96834×10^{-7}	9.48698×10^{-9}	6.78576×10^{-10}

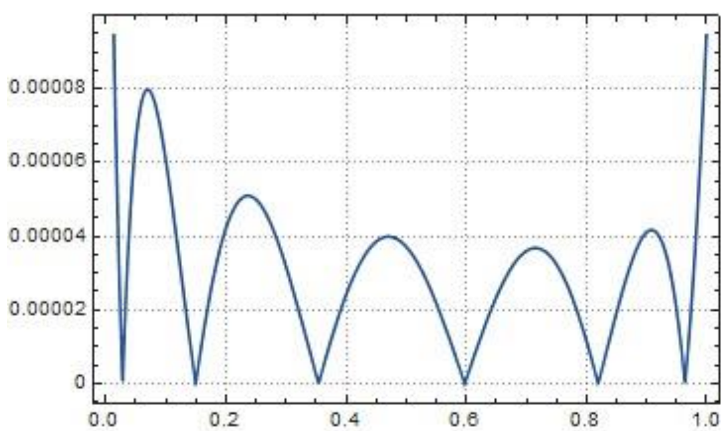

Figure 4: The absolute errors of $x(t)$ with m = 5 for Example 4.

Example 5. *Consider the following FOCP as [29]*

$$\min \quad J=\int_0^1 \left[e^t (x(t)-t^4+t-1)^2+(1+t^2)(u(t)+1-t+t^4-\frac{8000t^{\frac{21}{10}}}{77\Gamma(\frac{1}{10})})^2\right]dt \tag{23}$$

$s.t.$

$$\begin{cases} D^{1.9}x(t)=x(t)+u(t), & 0\le t\le 1, \\ x(0)=1, & \dot{x}(0)=-1. \end{cases} \tag{24}$$

For this problem $x(t)=1-t+t^4$, minimize the performance index $\tilde{J}$ is zero. In Table 5, the approximate values of minimum for different values of m are presented. Also, Fig. 5 demonstrates the absolute errors of $x(t)$ with m = 8.

Table 5:The comparison of estimated value of $\tilde{J}$ for different values of m,for Example5

m	***Method in* [29]**	***Our method***
4	5.42028×10^{-7}	5.36896×10^{-7}
5	6.77757×10^{-8}	6.73399×10^{-8}
7	2.84624×10^{-9}	2.82315×10^{-9}
8	8.22283×10^{-10}	6.88063×10^{-10}

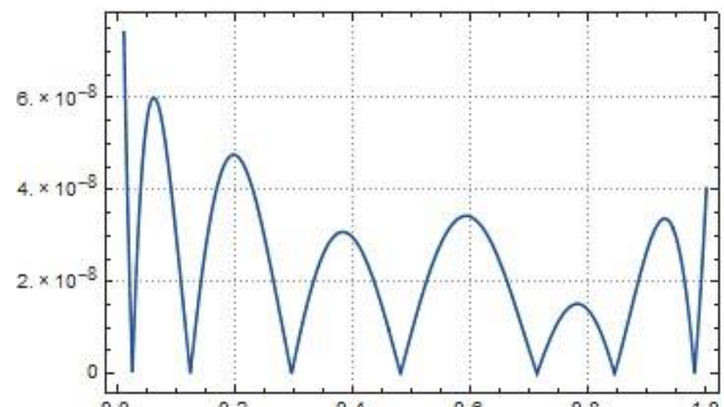

Figure 5: The absolute errors of $x(t)$ with m = 8 for Example 5.

6 CONCLUSION

In this paper, we have presented the alternative Legendre polynomials matrix of Riemann-Liouville fractional integration for the first time. The fractional optimal control problem is transformed into an equivalent variational problem, then the variational problem is solved approximately by utilizing the alternative Legendre polynomials basis, operational matrix of fractional integration, Gauss-Legendre integration formula and Newton's iterative method for solving the system of equations. Finally, some test problems are included to show the validity and applicability of the new method.

REFERENCES

R.L. Bagley, P.J. Torvik, A theoretical basis for the application of fractional calculus to viscoelasticity. J. Rheol. 27, 201−210 (1983).

R.T. Baillie, Long memory processes and fractional integration in econometrics. J. Econom. 73, 5−59 (1996).

F. Mainardi, Fractional calculus: some basic problems in continuum and statistical mechan- ics, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, New York: 291−348 (1997).

Y.A. Rossikhin, M.V. Shitikova, Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids. Appl. Mech. Rev. 50, 15−67 (1997).

R.L. Magin, Fractional calculus in bioengineering. Crit. Rev. Biomed. Eng. 32, 1−104 (2004).

T.S. Chow, Fractional dynamics of interfaces between soft-nanoparticles and rough sub- strates. Phys. Lett. A. 342, 148−155 (2005).

J. Sabatier, O.P. Agrawal, J.A. Tenreiro Machado, Advances in Fractional Calculus. Springer. (2007).

A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differen- tial Equations. in: North-Holland Mathematics Studies, 204, Elsevier Science B.V, Amster- dam (2006).

O.P. Agrawal, A general formulation and solution scheme for fractional optimal control problems. Nonl. Dyn. 38, 323−337 (2004).

O.P. Agrawal, Fractional optimal control of a disributed system using eigenfunctions. ASME. J. Comput. Nonl. Dyn. doi:10.1115/1.2833873, 6 pages (2008).

A. Lotfi, S.A. Yousefi, A numerical technique for solving a class of fractional variational problems. J. Comput. Appl. Math. 237, 633−643 (2013).

P. Rahimkhani, Y. Ordokhani, A numerical technique for solving fractional variational prob- lems by Müntz-Legendre polynomials, J. Appl. Math. Comput. DOI: 10.1007/s12190-017- 1134-z, (2017).

A. Lotfi, S.A. Yousefi, M. Dehghan, Numerical solution of a class of fractional optimal control problems via the Legendre orthonormal basis combined with the operational matrix and the Gauss quadrature rule. J. Comput. Appl. Math. 250, 143−160 (2013).

Ch. Tricaud, Y. Chen, An approximate method for numerically solving fractional order optimal control problems of general form. Comput. Math. Appl. 59, 1644−1655 (2010).

O.P. Agrawal, A formulation and numerical scheme for fractional optimal control problems. J. Vib. Control 14, 1291−1299 (2008).

P. Rahimkhani, Y. Ordokhani, E. Babolian, An efficient approximate method for solving delay fractional optimal control problems. Nonlinear Dyn. 86, 1649−1661 (2016).

N. Ozdemir, O.P. Agrawal, B.B. Iskender, B. Karadeniz, Fractional optimal control of a 2-dimensional distributed system using eigenfunctions. Nonl. Dyn. 55(3), 251−260 (2009).

P. Rahimkhani, Y. Ordokhani, Generalized fractional-order Bernoulli-Legendre functions: an effective tool for solving two-dimensional fractional optimal control problems, IMA Jour- nal of Mathematical Control and Information, doi: https://doi.org/10.1093/imamci/dnx041 (2017).

Y. Postenko, Time-fractional radial diffusion in sphere. Nonl. Dyn. 53, 55−65 (2008).

H. Qi, J. Liu, Time-fractional radial diffusion in hollow geometries. Meccanica 45(4), 577−583 (2010).

P. Rahimkhani, Y. Ordokhani, E. Babolian, Müntz-Legendre wavelet operational matrix of fractional-order integration and its applications for solving the fractional pantograph differential equations. Numer Algor., DOI: 10.1007/s11075-017-0363-4.

P. Rahimkhani, Y. Ordokhani, E. Babolian, Fractional-order Legendre wavelets and their ap- plications for solving fractional-order dierential equations with initial/boundary conditions. Computational Methods for Dierential Equations, 5(2), 117−140 (2017).

V.S. Chelyshkov, Alternative orthogonal polynomials and quadratures. Electron. Trans. Nu- mer. Anal. 25(7), 17−26 (2006).

S. Bazm, A. Hosseini, Numerical solution of nonlinear integral equations using alternative Legendre polynomials. J. Appl. Math. Comput. DOI: 10.1007/s12190-016-1060-5, (2016).

V.S. Chelyshkov, Alternative Jacobi polynomials and orthogonal exponentials. 55, arXiv: 1105.1838, (2011).

P. Rahimkhani, Y. Ordokhani, E. Babolian, Fractional-order Bernoulli wavelets and their applications. Appl. Math. Modell. 40, 8087− 8107 (2016)

E. Keshavarz, Y. Ordokhani, M. Razzaghi, A numerical solution for fractional optimal con- trol problems via Bernoulli polynomials. Journal of Vibration and Control, 22(18) (2016).

K. Rabiei, Y. Ordokhani, E. Babolian, The Boubaker polynomials and their application to solve fractional optimal control problems. Nonlinear Dyn, 88(2), 1013− 1026 (2017).

A. Lotfi, S.A. Yousefi, M. Dehghan, Numerical solution of a class of fractional optimal control problems via the Legendre orthonormal basis combined with the operational matrix and the Gauss quadrature rule. J. Comput. Appl. Math. 250, 143− 160 (2013).