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3- regular self-complementary 6- hypergraphs

O.Naserian
Department of Mathematics, Zanjan Branch, Islamic Azad University,
Zanjan, Iran
o.naserian@gmail.com

A. H. Shokouhi
Department of Information Technology, Payame Noor University,
Tehran, Iran
a.h.shokuhi@gmail.com

ABSTRACT

A k-hypergraph with vertex set V and edge set E is called t-regular if every t-element subset of V lies in the same number of elements of E. In this note, we prove the existence of a new family of 3-regular self-complementary k-hypergraphs for k=6.

KEYWORDS: k-hypergraph, self-complementary hypergraph, t-regular hypergraph

1 INTRODUCTION

Definition 1.1. A k-uniform hypergraph of order v is an ordered pair H = (V, E), where V = V (H) is a v-set (called vertex set) and E = E(H) is a subset of $P_k(V)$ (called edge set). We call a k-uniform hypergraph simply a k-hypergraph [8].

A k-hypergraph H of order v is t-subset-regular (for short t-regular) if there exists a positive integer (called the t-valence of H), such that each element of $P_t(V)$) is a subset of exactly λ elements of E(H). Henceforth, we denote such a structure by RHG(t, k, v). So RHG(t, k, v) is a t-(v,k, λ) design. Two k-hypergraphs H_1 and H_2 are isomorphic, if there is a bijection θ : V (H_1) → V (H_2), such that θ induces a bijection from E(H_1) into E(H_2).A k-hypergraph H is called self-complementary if H is isomorphic to H' = (V, $P_k(V)$ \ E(H)). An antimorphism of self-complementary hypergraph H, is an isomorphism between H and H'. If H is a self-complementary RHG(t, k, v), then H and H' form a large sets of t-designs LS[2](t, k, v) with an additional property that these two designs are isomorphic. Henceforth, we denote this structure by SRHG(t, k, v). An easy counting argument shows that an SRHG(t, k, v) is also an SRHG(i, k,

v) for $0 \leq i \leq t$. Hence a set of necessary conditions for the existence of an SRHG(t, k, v) is that $\binom{v-i}{k-i}$ is an even integer for all i = 0, 1, ..., t.

Much of the research to date into t-regular self-complementary k-hypergraphs has been focused on determining necessary and sufficient conditions on the order of these structures. The following theorem gives the necessary conditions in terms of some congruence relations. Let p be a prime number and r and m be positive integers. Then by $r_{[m]}$ we denote the remainder of division r by m and by $r_{(p)}$ we denote the largest integer i such that p^{i} divides r.

Theorem 1.2. [6] If there exists an SRHG(t, k, v), then there exists an integer q, where $k_{(2)} < q \leq \min\{i : 2^{i} > k\}$ such that

$v_{[2^q]} \in \{t, t+1, \ldots, k_{[2q]-1}\}$.

It should be noted that in [6] the above theorem is stated for large sets of t-designs. The necessary conditions of above theorem have been shown to be sufficient in the special cases. The case t =1 and k =2 was handled constructively by Rao[10], but there is also a proof due to Wilson [11]. Potočnik and Šajna handled the case where k =3 and t =1 [9], and Konr and Potočnik handled the case where k =3 and t =2 [8]. The next great achievement was obtained by Gosselin who showed that the necessary conditions of Theorem 1.2 are sufficient for all k, in the case where t=1 [4]. Recently, it was proved for t =2 and k =4, 5 [3]. In Section 3, we show that the necessary conditions are sufficient for t =3 and k = 4, 5.

Remark 1.3. Let H be an SRHG(t, k, v), θ be an antimorphism of H and x ∈ V be a fixed point of e. Now consider

$H^{d} = \{B - \{x\} \mid B \in H, x \in B \in H\}$,

$H^{r} = \{B \mid x \notin B \in H\}$,

$H^{c} = \{V - B \mid B \in H\}$.

Then one can easily see that H^{d} and H^{r} are SRHG(t − 1,k − 1,v − 1) and SRHG(t − 1,k,v − 1), respectively and in turn are called the derived and the residual hypergraphs of H with respect to x. Clearly H^{c} is also an SRHG(t, v − k, v) with e as an antimorphism, that is called the complement hypergraph of H [5].

We may obtain more hypergraphs from a given hypergraph as the following theorem suggests (see[7]). The proof is clear by successive applying of the above remark.

Theorem 1.4. If there exists an SRHG(t, k, v) with an antimorphism having at least t fixed points, then there exists SRHG(t − i, k − j, v − l) for all $0 \leq j \leq l \leq i \leq t$.

2 PARTITIONABLE SETS

A powerful method in constructing large sets is obtained from the notion of partitionable sets [1]. In what follows we generalize this method to construct hypergraphs with different parameters.

Let $H_1, H_2 \subseteq P_k(V)$. We say that H_1 and H_2 are t-equivalent if every t-subset of V appears in the same number of members of H_1 and H_2. If there exists a partition of $H \subseteq P_k(V)$ into N mutually t-equivalent subsets, then H is called an (N, t)-partitionable set. If H = $\{H_1, H_2\}$ is a Let V_1 and V_2 be two disjoint sets and $H_i \subseteq P_{k_i}(V_i)$ for i =1, 2. In what follows we need the following definition:

$$H_1 * H_2 = \{h_1 \cup h_2 | h_1 \in H_1, h_2 \in H_2\}.$$

In following the Lemma, we give a recursive method to construct SRHG(t, k, v).

Lemma 2.1. [3] Let a be a permutation on V . If there exist an SRHG(t, k, v) and an SRHG(t, k + 1,v) and a be their common antimorphism, then there exists an SRHG(t, k +1,v + 1).

Theorem 2.2. [3] Assume that there exist SRHG(t, i, v1) for all t +1 ≤ i ≤ k with θ_1 as an antimorphism such that each hypergraph of SRHG(t − 1, t, v_1 − 1) is (2, t_1)-partitionable set and also suppose there exists SRHG(t, k, v_2) such that θ_2 be an antimorphism, then an SRHG(t, k, $v_1$1 + v_2 − t) exists.

Let θ be a permutation on a v-set with at least t fixed points.

Corollary 2.3. [3] If there exist an SRHG(t, i, v) for t +1 ≤ i ≤ k with e as an antimorphism such that each hypergraph of SRHG(t−1, t, v−1) is (2,t)-partitionable set and also there exist SRHG(t, k, u) with an antimorphism having at least t fixed points, then there exist SRHG(t, k, u + l(v − t)) for all l ≥ 1.

Let θ be a permutation on (v + k)-set with at least (k − 1) fixed points.

Corollary 2.4. [3] If there exist SRHG(t, i, v + i) for t +1 ≤ i ≤ k with e as an antimorphism such that each hypergraph of SRHG(t − 1, t, v + k − 1) is (2,t)-partitionable set and also if there exist SRHG(t, k, u) with an antimorphism having at least k −1 fixed points, then there exist SRHG(t, k, u+ l(v + 1)) for all l ≥ 1.

3 MAIN RESULTS

Alltop[2] has proved a theorem on extending t-designs. We prove a similar result for t-regular self-complementary k-hypergraphs.

Theorem 3.1. If there exists an SRHG(t, k, 2k + 1), then there exists an SRHG(t +1,k +1, 2k + 2).

Proof. Let X be a (2k+1)-set and x ∉ X. Suppose that *{H*1, *H*2*}* is an SRHG(*t; k;* 2*k* + 1). For *i* = 1*;* 2, define

$$E_i = \{h \cup \{x\} | h \in H_i\},$$
$$F_i = \{X \setminus h | h \in H_i\}.$$

Clearly, E_ii and F_i partition P_{k+1} (X ∪{x}). We show that These sets are t-equivalent. Let x be a fixed point for antimorphism.

Example 3.2. SRHG(2, 5, 11) there exists[3]. Thus by Theorem 3.1, SRHG(3, 6, 12) there exists.

We use Corollary 2.3 to give some existence results on SRHG(3, k, v). At first step note to the following corollary of Theorem 1.2. This corollary presents a necessary condition to the existence of SRHG(3, k, v).

Corollary 3.3. Suppose that there exists an SRHG(3, k, v). Then
(i) If k = 4, then v ≡ 3 (mod 8);
(ii) If k =5, then v ≡3, 4 (mod 8).

Now we show that the necessary conditions for the existence of SRHG(3, k, v) for k = 4, 5 are sufficient.

Theorem 3.4. There exist an SRHG(3, 6,v) if and only if v ≡3, 4, 5 (mod 8).

Proof. We have to establish the existence of SRHG(3,6,11), SRHG(3,6,12), SRHG(3,6,13), each with an antimorphisms having at least 3 fixed points (note Corollary 2.3). For these hypergraphs, the first one exists by Remark 1.3 and the second one by Example3.2 exist. The last one exists by Lemma 2.1.

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