Strongly and Nicely Edge Distance-Balanced Graphs

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ABSTRACT
A nonempty graph $G$ is called nicely edge distance-balanced (NEDB), whenever there exists a positive integer $\gamma_G$, such that for any edge say $e = ab$ we have: $m_G(e) = m_G(e') = \gamma_G$. Which $m_G(e)$ denotes the number of edges laying closer to the vertex $a$ than vertex $b$ and $m_G(e')$ is defined analogously. Also, a nonempty graph $G$ is strongly edge distance-balanced, for every edge say $e = ab$ of $G$ and every $i \geq 0$ the number of edges at distance $i$ from $a$ and at distance $i+1$ from $b$ is equal to the number of edges at distance $i+1$ from $a$ and at distance $i$ from $b$. In this paper, we study on some properties of strongly edge distance-balanced graphs. Later, we discuss on some operations of graphs and at last by the help of definition of SEDB graph, classify the NEDB graphs with $\gamma'_G = 3$.

KEYWORDS: Graph, Diameter of graph, Strongly distance-balanced graph, Strongly edge distance-balanced graph, Nicely edge distance-balanced graph.

1 INTRODUCTION
Let $G$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. If $e = ab \in E(G)$, then $d_G(a, b)$ stands for the distance between $a$ and $b$ in $G$ and it means number of vertices which are lying in the shortest path between $a$ and $b$. Also, consider any two edges in $G$, say $e = ab$ and $e' = ab$, the distance between $e$ and $e'$ is defined as:

$$d_G(e') = \min\{d_G(e'), a, d_G(e), b\}.$$ 

The quantities $n_G(a, e), n_G(e)$ and $n_G(b, e)$ are defined to be the number of vertices equidistant from $a$ and $b$, the number of vertices whose closer to vertex $a$ than vertex $b$ and the number of vertices closer to $b$ than $a$, respectively. Similarly, the quantities $m_G(a, e), m_G(e)$ and $m_G(b, e)$ are defined to be the number of edges equidistant from $a$ and $b$ and the number of edges whose closer to vertex $a$ than $b$ and the number of edges closer to $b$ than $a$, respectively. Let $ab$ be an arbitrary edge of $G$. Then for any two non-negative integer $i$, $j$, we have:

$$D^{i,j}_G(e) = \{e' \in E(G) \mid d_G(e'), a = i, \ d_G(e'), b = j\}.$$ 

By the definition of NEDB, $D^{i,j}_G(e) = \emptyset$. The triangle inequality implies that only the sets $D^{i,j}_G(e)$, $D^{i+1,j}_G(e)$, for each $(2 \leq i \leq d + 1)$ must be nonempty.

Recall the definition of transmission, $T(a)$ of a vertex $u \in V(G)$ is defined as $T(a) = \sum_{v \in V(G)} d(u, b)$, see [10]. Also, consider $a$ as an arbitrary vertex in $G$, defined the edge-transmission $T(a)$ defined as $T(a) = \sum_{e \in E(G)} d(e, a)$.

The graph $G$ is called distance-balanced (as brief DB), if for any arbitrary edge $e = ab$ of $G$, the number of vertices are lying closer to $a$ than to $b$ is equal to the number of vertices which are lying closer to $b$ than to $a$, [5, 7, 10].

The simple connected $G$ is called strongly distance-balanced (SDB), if for any edge $e = ab$ in $G$ and any arbitrary integer $i$, the number of vertices at distance $i$ from $a$ and at distance $i+1$ from $b$ is equal to the number of vertices at distance $i+1$ from $a$ and at distance $i$ from $b$, [1, 9].
A nonempty graph \( G \) is called nicely distance-balanced (in short form NDB), whenever there exists a positive integer \( G \), such that for any two adjacent vertices \( a \) and \( b \) in \( G \), there are exactly \( G \) vertices of \( G \) which are closer to \( a \) than to \( b \), and exactly \( G \) vertices of \( G \) which are closer to \( b \) than to \( a \), see [11].

Edge distance-balanced graphs (as brief EDB), are such graphs in which for every edge \( e = ab \) the number of edges closer to vertex \( a \) than to vertex \( b \) is equal to the number of edges closer to \( b \) than to \( a \), [12]. In the other hand, one can easily find a graph \( G \) as an EDB graph, if and only if:

\[
mc_{ab}(e) = mc_{ba}(e), \text{ for any edge } e = ab \in E(G).
\]

The simple connected \( G \) is called strongly distance-balanced (SDB), if for any edge \( e = ab \) in \( G \) and any positive integer \( i \), the number of vertices at distance \( i \) from \( a \) and at distance \( i \) from \( b \) holds if and only if \( |Si(a)| = |Si(b)|, \) for \( i \in \{0, 1, 2, \ldots, d\} \) and every edge \( ab \) of \( G \), where

\[
Si(a) = \{ x \in V(G) \mid d_G(x, a) = i \}.
\]

By above definition, if \( |D_{i+1}(e)| = |D_i(e)|, \) for \( i \in \{1, 2, \ldots, d\} \), then \( G \) is edge distance-balanced. But the converse is not true.

### 2 Some properties of SEDB graphs

In this section, we study on some basic properties of strongly edge distance-balanced graphs and try to understand under which conditions we have SEDB graph.

**Proposition 2.1** If \( G \) be a connected and strongly edge distance-balanced graph, then \( G \) is regular.

**Proof.** Let \( G \) be a connected strongly edge distance-balanced graph. Let \( e = ab \in E(G) \), by definition of SEDB graph we have \( |D_i(e)| = |D_i'(e)| \). But we know that, \( |D_0'(e)| = \deg(a) - 1 \) and \( |D_0'(e)| = \deg(b) - 1 \). Thus, \( \deg(a) = \deg(b) \), for any \( a, b \in V(G) \). Hence the result.

**Proposition 2.2** Let \( G \) be a graph with diameter \( d \) and \( T(a) = \{ e \in E(G) \mid d_G(e, a) = i \} \). If \( G \) be strongly edge distance-balanced, then \( |T_i(a)| = |T_i(b)| \), for any edge \( e = ab \) in \( G \) and for \( i \in \{1, 2, \ldots, G\} \). The converse holds if \( G \) be a regular graph.

**Proof.** Let us assume that \( G \) is strongly edge distance-balanced and let \( ab \in E(G) \). By definition, we have \( |D_i(e)| = |D_i'(e)| \), for \( i \in \{1, 2, \ldots, d\} \). Since \( |T_i(ab)| = |T_i'(ab)| + |D_i(e)| + |D_i'(e)| \), and \( |T_i(e)| + |D_i'(e)| \). So \( |T_i(e)| = |T_i'(e)| \), for \( i \in \{1, 2, \ldots, d\} \). For next part, assume that \( G \) is regular. Using induction on \( i \), we now show that \( |D_i'(e)| = |D_i'(e)| \), holds for every edge say \( e = ab \in E(G) \), for any \( i \in \{1, 2, \ldots, d\} \). If \( i = 1 \), then \( |D_0'(e)| = \deg(a) - 1 \) and \( |D_0'(e)| = \deg(b) - 1 \) and \( G \) is regular, we have \( |D_0'(e)| = |D_0'(e)| \). By hypothesis of induction we have \( |D_{k+1}(e)| = |D_k'(e)| \), for \( 1 \leq k \leq d - 1 \). Observe that,

\[
|D_{k+1}(e)| = T_k(ab) + |D_{k+1}(e)| + |D_k(e)|, \\
|D_{k+1}(e)| = T_k(ab) + |D_{k+1}(e)| + |D_k(e)|.
\]

By \( |D_{k+1}(e)| = |D_{k+1}(e)| \), hence the result.

**Proposition 2.3** Let \( G \) be a connected graph with diameter \( d \). If \( G \) be strongly edge distance-balanced, then \( G \) is strongly distance-balanced.

**Proof.** Let \( G \) be a connected graph which is strongly edge distance-balanced and for any \( a, b \in V(G) \) and \( i \in \{1, 2, \ldots, d\} \), we define

\[
A_i = \{ u \in V(G) \mid d_G(u, v) = i \}, \\
B_i = \{ u \in V(G) \mid d_G(u, v) = \deg(u) \}.
\]

If \( a \in A_i \), then \( a \in S_i(a) \) and if \( a \in B_i \), then \( a \in S_i(a) \). Thus we have

\[
S_i(a) = A_i + |B_i| + C_i - C_i \cap B_i.
\]

Similarly, we have:

\[
S_i(b) = A_i + |B_i| + C_i - C_i \cap B_i.
\]

Since \( G \) is connected strongly edge distance-balanced graph, we have
\[ A^i = A^i, \quad B^i = B^i, \quad C^i = C^i, \quad C^i \cap B^i = C^i \cap B^i. \] Thus, \( S(a) = S(b) \). Hence the result.

The converse of the above theorem is not true, for example Generalized Petersen graph \( GP(7, 2) \) is strongly distance-balanced graph which is not strongly edge distance-balanced graph.

3 SEDB graph and graph products

In this paper, if \( G \) and \( H \) are two graphs, the vertex set of Cartesian Product of them is
\[ V(G \square H) = V(G) \times V(H) \] and \((x, y)(x', y')\) is an edge of \( G \square H \), if \( x = x' \) and \( yy' \in E(H) \) or \( xx' \in E(G) \) and \( y = y' \).

**Proposition 3.1** Let \( G \) and \( H \) be strongly edge distance-balanced as well as vertex distance-balanced graphs. Then \( G \square H \) is strongly edge distance-balanced graph.

**Proof.** Let us assume that the below partition of \( E(G \square H) \):
\[ A = \{(a, x)(b, y) \in E(G \square H) \mid ab \in E(G), x = y\}, \]
\[ B = \{(a, x)(b, y) \in E(G \square H) \mid xy \in E(G), a = b\}. \]
Again, assume that \( G \) and \( H \) are strongly vertex and edge distance-balanced graphs and \( (a, x)(b, y) \in A \), for any \( i \in \{0, 1, 2, \ldots, d\} \), in graph \( G \square H \) we have,
\[ |D'_{i,1}(a, x)(b, y)| = |D'_{i,1}(e)| \cdot |E(H)| + |D'_{i,1}(e)| \cdot |V(H)|, \]
\[ |D'_{i,1}(a, x)(b, y)| = |D'_{i,1}(e)| \cdot |E(H)| + |D'_{i,1}(e)| \cdot |V(H)|. \]
In same way we have:
\[ |D'_{i,1}(a, x)(b, y)| = |D'_{i,1}(e)| \cdot |E(H)| + |D'_{i,1}(e)| \cdot |V(H)|, \]
\[ |D'_{i,1}(a, x)(b, y)| = |D'_{i,1}(e)| \cdot |E(H)| + |D'_{i,1}(e)| \cdot |V(H)|. \]
Therefore,
\[ |T^{G \square H}_{i,1}(a, x)| = S(a) \cdot |E(H)| + |T(a)| \cdot |V(H)|, \]
\[ |T^{G \square H}_{i,1}(b, y)| = S(b) \cdot |E(H)| + |T(b)| \cdot |V(H)|. \]
Since \( G \) is strongly vertex and edge distance-balanced, we have \( |S(a)| = |S(b)| \) and \( |T(a)| = |T(b)| \). Therefore \( |T^{G \square H}_{i,1}(a, x)| = |T^{G \square H}_{i,1}(b, y)| \).
Similarly, this result is going to be true for any arbitrary edge say \( e = (a, x)(b, y) \) in \( B \). Hence the result.

Let \( G \) and \( H \) be two graphs. The corona product \( G \ast H \) is obtained by taking one copy of \( G \) and \( |V(G)| \) copies of \( H \), and by joining each vertex of the \( i \)-th copy of \( H \) to the \( i \)-th vertices of \( G \), \( i = 1, 2, \ldots, |V(G)| \).
By the help of definition, every strongly edge distance-balanced graph is edge distance-balanced graph. Also, by [1] next result is clear.

**Proposition 3.2** The corona product of any two nontrivial, connected graphs is not strongly edge distance-balanced.

**Theorem 3.1** If \( G \) is connected and has diameter 2, then the following statements are equivalent:
\( a \) \( G \) is edge distance-balanced,
\( b \) \( G \) is strongly edge distance-balanced,
\( c \) \( G \) is regular.

**Proof.** Condition \( (b) \) and \( (c) \) are equivalent for graphs with diameter 2. Since \( G \) has diameter 2, for any vertex \( b \in V(G) \), we have \( D_2(b, G) = |E(G)| \cdot \deg(b) \), where \( D_2(b, G) = \sum_{e \in E(G)} d(G, e, b) \). Thus, \( \deg(a) = \deg(b) \) if and only if \( D_t(a, G) = D_t(b, G) \). It was proved in 5 that \( G \) is edge-distance-balanced if and only if for every \( a, b \in V(G) \), \( D_t(a, G) = D_t(b, G) \). Thus the equivalent \( (a) \) and \( (c) \) follows.
4 Classification

**Theorem 4.1** A graph $G$ is NEDB graph with $\gamma' G = 3$, if and only if it is one of the following graphs:

i) the complete bipartite graph $K_{4,4}$,

ii) the complete graph $K_5$,

iii) the Johnson graph $J(5, 1) \approx$ complete graph $K_5$,

iv) the Generalized Petersen $GP(3, 1) \approx GP(3, 2)$,

v) multipartite graph $K_{3 \times 2}$.

**Proof.** Let consider all possible cases for $\gamma' G = 3$. By [Proposition 2.2], $d \leq \gamma' G = 3$. On the other hand, $d$ can get 0, 1, 2 or 3. The result for $d = 0$ is clear. Now assume other cases:

First case: If $d = 1$, then $G$ is a complete graph, so $\gamma' G = n - 2$, since $\gamma' G = 3$ which means a complete graph on 5 nodes, so $G$ is $K_5$, which is congruent to $J(5, 1)$, hence the proof for (iv).

Second case: If $d = 2$ then we can consider two subcases:

Subcase 1: $D^{D_2}(e) = \phi$.

First if we assume $D^{D_3}(e) = \phi$, then we conclude $\Sigma_{i=1}^{d+1} D_i^{\gamma}(e) = 0$. By using Proposition 1.1 and $\gamma' G = 3$, so number of edges in $G$ must be 7. Now, let us consider the cases which may occurs: If $|D^{D_2}(e)| = |D^{D_2}(e)| = 3$, then $D^{i+1}(e)$ or $D'^{i+1}(e)$ must be empty. So we have a tree which is not NEDB.

If $|D^{D_2}(e)| = |D^{D_2}(e)| = 2$, since $\gamma' G = 3$, so $|D^{D_2}(e)| = |D^{D_2}(e)| = 1$. This graph is possible when these two edges in $D^{D_2}(e)$ and $D^{D_2}(e)$ are adjacent, (O.W. it is contradiction to $D^{D_3}(e) = \phi$). By 6 these assumptions, graph is not regular so it cannot be NEDB, which is a contradiction to Proposition 2.3. Here, $|D^{D_2}(e)| = 1$, if $|D^{D_2}(e)| = |D^{D_2}(e)| = 3$ then $G$ is a tree and it is irregular.

If $|D^{D_2}(e)| = |D^{D_2}(e)| = 2$ then $|D^{D_2}(e)| = |D^{D_2}(e)| = 1$. Since, $\gamma' G = 3$, here we have two vertices of degree 3 and the rest 4 have degree 2, which is not regular graph. At last, consider $|D^{D_2}(e)| = |D^{D_2}(e)| = 1$, again graph is irregular.

Subcase 2: $D^{D_2}(e) = \phi$.

Also, assume $D^{D_3}(e) = \phi$. Again, if we consider $|D^{D_2}(e)| = |D^{D_2}(e)| = 3$, we get tree, which is contradiction to NEDB. The all remaining cases as above are irregular except when $|D^{D_2}(e)| = 2$ and $|D^{D_2}(e)| = |D^{D_2}(e)| = 2$ and $|D^{D_2}(e)| = |D^{D_2}(e)| = 1$ then we get $GP(3, 1)$, which satisfies (v).

By same argument, if $|D^{D_2}(e)| = |D^{D_2}(e)| = 3$, one can see for only $|D^{D_2}(e)| = 5$ and $|D^{D_2}(e)| = 9$, we have 6 and 8 vertices so the graphs are $K_{3 \times 2}$ and $K_{4 \times 4}$, respectively. Hence, (iii) and (iv).

Third case: $d = 3$.

Subcase 1: $D^{D_2}(e) = \phi$. First, consider $D^{D_3}(e) = \phi$, so $|D^{D_1}(e)| = 0$. Suppose $|D^{D_1}(e)| = |D^{D_2}(e)| = 1$ and $|D^{D_3}(e)| = |D^{D_3}(e)| = 1$. Because $\gamma' G = 3$ and here $d = 3$ so $|D^{D_4}(e)| = |D^{D_4}(e)| = 1$, which is $C_7$.

Hence the proof of (i).

By considering same subcases as before, we can observe that all the other cases are irregular, which are not NEDB.

Forth case: $d = 4$.

Same as before, we can consider two subcases: and if $|D_1^{D_2}(e)| = |D_1^{D_2}(e)| = 1$ and $|D_3^{D_2}(e)| = |D_3^{D_2}(e)| = 1$ and $|D_5^{D_2}(e)| = |D_5^{D_2}(e)| = 1$, then we have a cycle on 8 nodes. This is the proof of (ii).

1) $D^{D_2}(e) = \phi$ and $D^{D_2}(e) \neq \phi$. By same argument, the only possible case occurs when $D^{D_2}(e) = \phi$. By considering same subcases as before, we can observe that all the other cases are irregular, which are not NEDB.

Forth case: $d = 4$.

Same as before, we can consider two subcases:

1) $D^{D_2}(e) = \phi$ and 2) $D^{D_2}(e) \neq \phi$.

By same argument, the only possible case occurs when $D^{D_2}(e) \neq \phi$ and if $|D_1^{D_2}(e)| = |D_1^{D_2}(e)| = 1$ and $|D_3^{D_2}(e)| = |D_3^{D_2}(e)| = 1$ and $|D_5^{D_2}(e)| = |D_5^{D_2}(e)| = 1$, then we have a cycle on 8 nodes.

This is the proof of (ii).
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CONCLUSION
The most important conclusion of this paper is the classification of Nicely Edge Distance-
Balanced Graphs according to $\gamma'_G =3$. By this assumption we can classify the graphs and also by
more calculation we can continue this work for $\gamma'_G \geq 4$ in next work.