3TH INTERNATIONAL CONFERENCE ON COMBINATORICS, CRYPTOGRAPHY AND COMPUTATION

Iran University of Science &Technology

Proceedings of the 3rd International Conference on Combinatorics, Cryptography and Computation (I4C2018)

Metric Dimension of Edge Corona

Irandokht Rezaee Abdolhosseinzadeh,
Department of Applied Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran.
ir_rezaee899@um.ac.ir;

Freydoon Rahbarnia
Department of Applied Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Iran.
Rahbarnia@um.ac.ir

ABSTRACT

The edge corona product of two graphs G and H, $G \diamond H$, is a graph obtained by taking a copy of G and $|E(G)|$ copies of H and joining each end vertex of i-th edge of G to every vertex in the i-th copy of H. In this paper, we define the graph $G \diamond^k H$ recursively from $G \diamond H$ as $G \diamond^k H = \left(G \diamond^{k-1} H\right) \diamond H$ and several results on the metric dimension of edge corona of graphs are obtained.

KEYWORDS: Edge Corona product; Metric dimension; Resolving set.

1 INTRODUCTION

Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ be a simple connected graph. The distance $\mathrm{d}(\mathrm{u}, \mathrm{v})$ between two vertices u and v is the lenght of shortest path between them. The diameter of G, $\mathrm{D}(\mathrm{G})$, is the greatest distance between any pair of vertices of G . The notation $\mathrm{u} \sim \mathrm{v}$, means that u and v are adjacent. For an ordered set $\mathrm{S} = \{\mathrm{r}_1, \dots, \mathrm{r}_k\}$ of vertices and a vertex v in a connected graph G , the ordered k-vector $\mathrm{r}(\mathrm{v}|\mathrm{S}) = (\mathrm{d}(\mathrm{v}, \mathrm{r}_1), \dots, \mathrm{d}(\mathrm{v}, \mathrm{r}_k))$ is called the representation of v with respect to S. The set S is called a resolving set for G , if distinct vertices of G have distinct representations with respect to S. The metric dimension of G, $\dim(\mathrm{G})$, is the minimum cardinality of any resolving set for G . The concepts of resolvability have some applications in chemistry for representing chemical compounds (Jonson, 1993) or to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures (Melter et al., 1984). For graphs representing networks, a resolving set represents a set of detecting devices in a network so that for every station in the network, there are two detecting devices whose distances from the station are distinct. Since it is important that the devices be properly maintained and have easy access to one another, it is convenient if these devices are located in close proximity to one another. Other applications of this concept to navigation of robots in networks and other areas appear in (Chartrand et al., 2000), (Khuller et al., 1996) and (Hulme et al., 1984). Garey and Johnson (Garey, 1979) showed that determining the metric dimension of an arbitrary graph is an NP-complete problem. Nevertheless, it is rather straightforward for some particular cases, such as those displayed in the Table 1, (see (Chartrand et al., 2000 and Tomescu et al., 2007) (for more details).)

Recall that C_n, K_n and P_n denote the cycle, the complete graph and the path of order n, respectively. A wheel graph W_n, is a graph formed by connecting a single vertex to all vertices of a cycle C_n. $K_{s,t}$ denotes a complete bipartite graph of order $s+t$. Q_n is hypercube graph with 2^n vertices and J_{2n} is Jahangir graph, which is obtained from the wheel W_{2n} by alternately deleting n spokes.

Lemma 1.1. The table below shows the metric dimension of some graphs.

Table 1 Metric dimension of some graphs.

G	P_n	C_n	K_n	Q_n	$K_{\{s,t\}}$	W_n	J_{2n}
$\|V(G)\|$	$n \geq 1$	$n \geq 3$	$n \geq 2$	2^n	$s+t \geq 3$	$n+1=4$	$n \geq 4$
$dim(G)$	1	2	$n-1$	n	$s+t-2$	3	$\left\lfloor \frac{2n}{3} \right\rfloor$

There are also some results of the metric dimension under graph operations (Caceres et al., 2005). The edge corona product of two graphs G and H, $G \diamond H$, is a graph obtained by taking a copy of G and $|E(G)|$ copies of H and joining each end vertex of i-th edge of G to every vertex in the i-th copy of H, (Hou et al.,2010 and Luo et al., 2018). Recently, Rinurwati et al. (2016) have provided the locally metric dimensions of edge corona product. Yero et al. (2011) have studied the metric dimension of corona product graphs, $G \circ H$. Motivated by their papers, we define the graph $G \diamond^k H$ recursively from $G \diamond H$ as $G \diamond^k H = \left(G \diamond^{k-1} H\right) \diamond H$ and also find some results on the metric dimension of $G \diamond^k H$.

2 MAIN RESULTS

General results of the metric dimensions of edge-corona product of two graphs will be presented in this section. Remember that, a block of G is a maximal nonseparable subgraph of G . Every two distinct blocks of G have at most one vertex in common; and if they have a vertex in common, then this vertex is a cut-vertex of G . Clearly, none vertices of H is a cut vertex in $G \diamond H$ and if a vertex $v \in V(G)$ is a cut vertex of G , then it is also a cut vertex of $G \diamond H$. Therefore, if G is a block, then $G \diamond H$ is a block. Moreover, if a graph G with $|V(G)| \geq 3$ is a block then $\delta \geq 2$.

lemma 2.2.

Let G be a block of order $n \geq 4$. Then for every pair of adjacent edges e_i and e_j, there exists an edge e_k such that e_k is adjacent to the edge e_i and is not adjacent to e_j, and also there exists an edge e_l such that e_l is adjacent to the edge e_j and is not adjacent to e_i.

Proof

Suppose to contrary, for two adjacent edges e_i and e_j, all other edges are adjacent to both e_i and e_j, since $n \geq 4$, the graph G should be isomorphic to one of the graphs in Figure 1. Which existence of a cut vertex v , is contradiction. ■

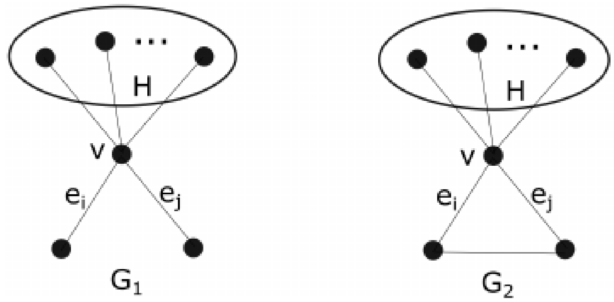

Figure 2: The subgraph H in G_1 and G_2 is an arbitrary graph.

Lemma 2.3.
For a graph $G = (V, E)$ of size m_1 and order $n_1 \geq 2$, and a graph H of order at least two, if $H_i = (V_i, E_i)$ is a copy of H corresponding to the edge $e_i = uv$ in $G \diamond H$, then

)1) For every two vertices u and v belong to V_i and the vertex $x \notin V_i$, we have $d(u, x) = d(v, x)$.
)2) If S is a resolving set for $G \diamond H$, then $S \cap V_i \neq \emptyset$ for every $i \in \{1, 2, \ldots, m_1\}$.
)3) If S is a resolving set for $G \diamond H$, then for every $i \in \{1, 2, \ldots, m_1\}$, $S \cap V_i$ is a resolving set for H_i.

Proof.
)1) Let $e_i = v_i v_{i+1} \in E$ and $x \notin V_i$. Without loss of generality, suppose $d(x, v_i) \leq d(x, v_{i+1})$, so by definition of edge corona product, we have

$$d_{G \diamond H}(u, x) = d_{G \diamond H}(u, v_i) + d_{G \diamond H}(v_i, x)$$
$$= 1 + d_{G \diamond H}(v_i, x) = d_{G \diamond H}(v, v_i) + d_{G \diamond H}(v_i, x) = d_{G \diamond H}(v, x).$$

)2) Suppose to the contrary, $S \cap V_i = \emptyset$ for some $i \in \{1, 2, \ldots, m_1\}$. Let $u, v \in V_i$, (1) implies that $d(u, x) = d(v, x)$ for all $x \in S$, so $r(u|S) = r(v|S)$ is a contradiction.
)3) Let $S_i = S \cap V_i$. For $u \in S_i$ or $v \in S_i$, it is clear to see that u and v have different representations with respect to S_i. Let $u, v \in V_i - S_i$, by taking into account that S is a resolving set for $G \diamond H$ and considering (1), the result is straightforward.■

Lemma 2.4.
Let $G = (V, E)$ be a block of order $n_1 \geq 4$ and size m_1. Let H be a connected graph of order at least two. If $H_i = (V_i, E_i)$ be a copy of H corresponding to the edge $e_i = uv$ in $G \diamond H$, then, if S is a minimum resolving set for $G \diamond H$, then $V \cap S = \emptyset$.

Proof.
We show that $S' = S - V$ is a resolving set for $G \diamond H$. Let u and v be two different vertices of $G \diamond H$, we have four cases as follows.

)1) $u, v \in V_i$. By Lemma **2.3** (2) there is a vertex $x \in V_i \cap S$ such that $d(u, x) \neq d(v, x)$. Clearly, $x \in S'$.
)2) $u, v \in V$. If u and v are not adjacent, then since G is connected there is a vertex w which is adjacent to u . Let $e_i = uw$. By Lemma **2.3** (2) there exists a vertex x, where $x \in V_i \cap S$ such that $d(u, x) = 1 < d(v, x)$. Since $V_i \cap S \subseteq S'$, then $x \in S'$. Now, let u and v be adjacent. Since $n_1 \geq 4$ there is a vertex w which is adjacent to v . Let $e_j = vw$ and let $y \in S \cap V_j$ then $d(v, y) = 1$ whereas $d(u, y) = d(u, v) + d(v, y) = 2$.
)3) $u \in V_i, v \in V_j, i \neq j$. If e_i is not adjacent to e_j, then by Lemma **2.3** (2) there exists a vertex x\in $V_i \cap S$ such that $d(u, x) \leq 2 < 3 \leq d(v, x)$ and $V_i \cap S \subseteq S'$ result in $x \in S'$, unless, by Lemma **2.2**, there is

an edge e_k adjacent to e_i which is not adjacent to e_j. By Lemma **2.3** (2), there is a vertex $z \in S \cap V_k$ such that $d(z,u) = 2 \neq 3 = d(z,v)$. Since $S \cap V_k \subseteq S'$, then $z \in S'$.
)4) $u \in V_i, v \in V$. If $vw = e_i$, then since $deg_G(v) \geq 2$, there is a vertex $t \in V$, $t \neq w$, such that $t \sim v$. Let $e_j = vt$. By Lemma **2.3** (2) let $x \in V_j \cap S$. So $d(v,x) = 1 < d(u,v) + 1 = 2$.
For $u \nsim v$, we take $x \in V_l \cap S$ where $e_l = vw$, in this case $d(v,x) = 1$ whereas $d(u,x) = d(u,v) + 1 > 1$.
Hence, S' is a resolving set for $G \diamond H$ and the proof is completes.■

Theorem 2.5.
Let G be a block of order $n_1 \geq 4$, size m_1, and let H be a connected graph of order $n_2 \geq 2$, size m_2. Then, $dim(G \diamond^k H) \geq m_1(1 + m_2 + 2n_2)^{k-1} \, dim\,(H)$,

Proof
If S is a minimum resolving set for $G \diamond H$, then by Lemma **2.4**, we have $S \cap V = \emptyset$. Furthermore, Lemma **2.3** (2) shows that for every $i \in \{1, \dots, m_1\}$, there exists a nonempty set $S_i \subseteq V_i$ such that $S = \bigcup_{i=1}^{m_1} S_i$. Now by using Lemma **2.3** (3), S_i is a resolving set for H_i. Therefore,
$dim(G \diamond H) = |S| = \sum_{i=1}^{m_1} |S_i| \geq \sum_{i=1}^{m_1} dim(H_i) = m_1 \, dim(H)$,
then the result is evidently true.■

Theorem 2.6.
Let $G = (V,E)$ be a block of order $n_1 \geq 4$ and size m_1 and let H be a connected graph of order $n_2 \geq 2$ and $H_i = (V_i, E_i)$ be the subgraph of $G \diamond H$ corresponding to the i-th copy of H. If $D(H) \leq 2$, then,
$dim(G \diamond^k H) = m_1(1 + m_2 + 2n_2)^{k-1} \, dim\,(H)$.

Proof
Let $S_i \subseteq V_i$ be a resolving set for H_i, we will show that if $D(H) \leq 2$, then $S = \bigcup_{i=1}^{m_1} S_i$ is a resolving set for $G \diamond H$. For two vertices $x, y \in V(G \diamond H)$, we have

)1) $x, y \in V_i$. Since $D(H) \leq 2$, $r(x|S) \neq r(y|S)$ follows from $r(x|S_i) \neq r(y|S_i)$.
)2) $x \in V_i$ and $y \in V$. There exists a vertex $z \in V$ which is adjacent to y.
Let $e_j = yz$, $j \neq i$. For every vertex $v \in S_j$, without loss of generality, suppose $d(y,x) \leq d(z,x)$. So $d(v,y) = 1 < d(v,y) + d(y,x) = d(v,x)$.
)3) $x \in V_i$ and $y \in V_j$, $i \neq j$. Let $v \in S_i$. Clearly, $d(v,x) < d(v,y)$.
)4) $x, y \in V$. Let $z \neq y$ be a vertex of G adjacent to x and let $e_i = xz$. Without loss of generality, assume that $d(x,y) \leq d(z,y)$. So, for $v \in S_i$, we have $d(v,x) = 1 < d(v,x) + d(x,y) = d(v,y)$.
Hence for every two distinct vertices x and y of $G \diamond H$, $r(x|S) \neq r(y|S)$. Therefore, $dim(G \diamond H) \leq m_1 \, dim\,(H)$ and also we have

$$dim(G \diamond^k H) \leq m_1(1 + m_2 + 2n_2)^{k-1} \, (n_2 - 2).$$

By Theorem **2.5** we conclude the result. ■
In order to show a consequence of the above theorem we present the following lemma, where N_t is an empty graph of order t.

Lemma 2.7.(chartrand et al., 2000)
Let G be a connected graph of order $n \geq 4$. Then $dim(G) = n - 2$, if and only if $G = K_{s,t}$, $(s, t \geq 1)$, $G = K_s + N_t$, $(s \geq 1,\ t \geq 2)$ or $G = K_s + (K_1 \cup K_t)$, $(s, t \geq 1)$.

By considering Lemma **2.7,** we obtain the following result.

Corollary 2.8.

Let G be a block of order $n_1 \geq 4$ and H be a graph of order $n_2 \geq 4$ and $D(H) \leq 2$. Then $dim(G \diamond^k H) = m_1(1 + m_2 + 2n_2)^{k-1}(n_2 - 2)$, if and only if $G = K_{s,t}$, $(s, t \geq 1)$, $G = K_s + N_t$, $(s \geq 1, t \geq 2)$ or $G = K_s + (K_1 \cup K_t)$, $(s, t \geq 1)$.

Now, by considering Lemma **1.1** and using Theorems **2.5** and **2.6**, the following result obtained.

Corollary 2.9. We have

1) $dim(C_n \diamond C_m) \geq 2n$.
2) $dim(C_n \diamond P_m) \geq n$.
3) $dim(K_n \diamond C_m) \geq n(n-1)$.
4) $dim(K_n \diamond P_m) \geq \frac{n(n-1)}{2}$.
5) $dim(Q_n \diamond K_m) = 2^{n-1}n\,(m-1)$.
6) $dim(K_{s,t} \diamond K_m) = st\,(m-1)$.
7) If $n \geq 4$ then, $dim(J_{2n} \diamond K_m) = 2^{n-1}n\,(m-1)$.

3 REFERENCES

J. Caceres, C. Hernando, M. Mora, M. L. Puertas, I. M. Pelayo, C. Seara, and D. R. Wood, (2005), On the metric dimension of some families of graphs, Electronic Notes in Discrete Math. 22, 129- 133.

I. G. Yero, D. Kuziak, and J. A. Rodriguez-Velazquez, (2011), On the metric dimension of corona product graphs, computers and mathematics with applications 61, 2793-2798.

Y. Hou, and W-C. Shiu, (2010), The spectrum of the edge corona of two graphs, Electron. J. Linear Algebra 20 , 586-594.

Rinurwati, Slamin, and H. Suprajitno, (2016), General results of local metric dimensions of edge-corona of graphs, International Mathematical Forum 11 (16) 793-799.

G. Chartrand, L. Eroh, M. A. Johnson, and O. R. Oellermann,(2000), Resolvability in graphs and the metric dimension of a graph, Discrete Applied Mathematics 105 , 99-113.

M. R. Garey, and D. S. Johnson, Computers and Intractibility: A Guide to the Theory of NP Completeness, W. H. Freeman and Company, (1979).

B. L. Hulme, A. W. Shiver, and P. J. Slater, (1984), A Boolean algebraic analysis of fire protection, Algebraic and Combinatorial Methods in Operations Research 95 215-227.

M. A. Johnson, (1993), Structure-activity maps for visualizing the graph variables arising in drug design, Journal of Biopharmaceutical Statistics 3 203-236.

S. Khuller, B. Raghavachari, and A. Rosenfeld, (1996), Landmarks in graphs, Discrete Applied Mathematics 70 217-229.

Y. Luo, and W. Yan, Spectra of the generalized edge corona of graphs, Discrete Math. Algorithm. Appl. 10, 1850002 (10 pages) (2018).

R. A. Melter and I. Tomescu, (1984), Metric bases in digital geometry, Computer Vision Graphics and Image Processing 25 113-121.

I. Rezaee Abdolhosseinzadeh, F. Rahbarnia, M. Tavakoli, and A. R. Ashrafi, (2017) Some vertex-degree-based topological indices under edge corona product, Italian J. of Pure and Applied Mathematics 38 , 81-91.

I. Tomescu, and I. Javaid, (2007), On the metric dimension of the Jahangir graph, Bull. Math. Soc. Sci. Math. Roumanie 4 371-376.