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Star-critical Ramsey number of matchings versus a complete graph

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ABSTRACT

For given graphs $G_1, \dots, G_t$, the *Ramsey number* $r(G_1, \dots, G_t)$ is defined as the smallest positive integer n such that $K_n \rightarrow (G_1, \dots, G_t)$ and the star-critical Ramsey number $r^*(G_1, \dots, G_t)$ is defined to be $\min\{\delta(F): F \subseteq K_r, F \rightarrow (G_1, \dots, G_t)\}$. The aim of this paper is to study the star-critical Ramsey number of matchings versus a complete graph. In fact, the exact value of the star-critical Ramsey number $r^*(K_n, m_1K_2, \dots, m_tK_2)$ will be computed for $n \geq t+1$.

KEYWORDS: Ramsey number, Star-critical, Free coloring, Matchings, Complete graph.

1 INTRODUCTION

In this note, we are only concerned with undirected simple finite graphs and we follow [1] for terminology and notations not defined here. For a graph G, we denote its vertex set, edge set, maximum degree, minimum degree and the complement graph of G by $V(G)$, $E(G)$, $\Delta(G)$, $\delta(G)$ and $\overline{G}$, respectively. If $v \in V(G)$, we use $\deg(v)$ and $N(v)$ to denote the degree and the set of neighbourhoods of v in G, respectively. Let U and V be two subsets of vertex set $V(G)$, by $E[U, V]$ we mean the set of all edges between U and V and we use $G[V]$ to denote the induced subgraph on vertex set V from graph G. In addition, for a k-edge coloring of a graph G with colors $\alpha_1, \dots, \alpha_k$, we use G^{α_i}, $1 \leq i \leq k$, to denote the spanning subgraph of G induced by the edges of color α_i. The *star* graph on $n+1$ vertices is denoted by $K_{1,n}$ and a complete graph on n vertices is denoted by K_n. A *clique* in a graph is a subset of vertices such that the induced subgraph on this vertices is a complete graph and we denote by $K_k(n_1, \dots, n_k)$ the complete k-partite graph in which the i-th part has n_i vertices, for $1 \leq i \leq k$. Also by a *stripe* of size m, mK_2, we mean a graph on $2m$ vertices and m independent edges.

For given graphs $G, G_1, G_2, \dots, G_t$, we write $G \rightarrow (G_1, G_2, \dots, G_t)$ if the edges of G are colored in any fashion with t colors, then for some j, $1 \leq j \leq t$, the spanning subgraph whose edges are colored with the j-th color, contains a copy of G_j. For given graphs $G_1, G_2, \dots, G_t$ the *Ramsey number* $r(G_1, G_2, \dots, G_t)$ is defined as the smallest positive integer n such that $K_n \rightarrow (G_1, G_2, \dots, G_t)$. The existence of such a positive integer is guaranteed by the Ramsey's classical result [7]. For a survey on Ramsey theory and results in this area, we refer the reader to the regularly updated survey by Radziszowski [6].

For given graphs $G_1, G_2, \dots, G_t$ with $r = r(G_1, G_2, \dots, G_t)$, a $(G_1, G_2, \dots, G_t)$-free coloring is defined as an edge coloring of K_{r-1} with t colors such that for each i, $1 \leq i \leq t$, the i-th color class contains no monochromatic copy of G_i. The star-critical Ramsey number $r^*(G_1, G_2, \dots, G_t)$ is defined to be

$$r^*(G_1, G_2, \dots, G_t) = \min\{\delta(F): F \subseteq K_r, F \to (G_1, G_2, \dots, G_t)\}.$$

The concept of the star-critical Ramsey number was first defined by Hook and Isaak in [3]. Let $K_{1,k}$ be a star and $K_n \sqcup K_{1,k}$ be a graph obtained by adding a new vertex v adjacent to k vertices of K_n. It is easy to see that for $r = r(G_1, G_2, \dots, G_t)$, the star-critical Ramsey number $r^*(G_1, G_2, \dots, G_t)$ is equivalent to the following definition.

$$r^*(G_1, G_2, \dots, G_t) = \min\{k | K_{r-1} \sqcup K_{1,k} \to (G_1, G_2, \dots, G_t)\}.$$

A class of given graphs $G_1, G_2, \dots, G_t$ with $r = R(G_1, G_2, \dots, G_t)$ is called Ramsey-full if $K_r \to (G_1, G_2, \dots, G_t)$, but for each edge $e \in K_r$, $K_r - e \nrightarrow (G_1, G_2, \dots, G_t)$. Therefore, $(G_1, G_2, \dots, G_t)$ is Ramsey-full if and only if $r^*(G_1, G_2, \dots, G_t) = r(G_1, G_2, \dots, G_t) - 1$.

Star-critical Ramsey numbers have been investigated by several authors (See for instance [5],[3]). In particular, it is proved that for integers $r \geq s \geq 1$, $r^*(rK_2, sK_2) = s$ and for integers $m \geq 1$ and $n > 2$, $r^*(K_n, mK_2) = n + 2m - 3$. In this note, we study the star-critical Ramsey number of a complete graph versus any number of matchings as follows.

Theorem 1.1 *Let $n \geq 3$, $t \geq 2$, $n \geq t + 1$ and $m_1, \dots, m_t$ be positive integers such that $m_1 = max\{m_1, \dots, m_t\}$. Then we have*

$$r^*(K_n, m_1K_2, \dots, m_tK_2) = \begin{cases} \sum_{i=1}^{t-1} (2m_i - 1) + 1 & n = t + 1 \\ \sum_{i=1}^{t} 2(m_i - 1) + n - 1 & n \geq t + 2 \end{cases}$$

2 PROOF OF THE MAIN RESULT

First, we start with the following theorem, giving the exact value of the Ramsey number of matchings versus a complete graph.

Theorem 2.1 ([4]) *Let $n \geq 3$, $m_1, \dots, m_t$ be positive integers and $m_1 = max\{m_1, \dots, m_t\}$, then*

$$r(K_n, m_1K_2, \dots, m_tK_2) = \begin{cases} \sum_{i=1}^{t} 2(m_i - 1) + n & n > t \\ \sum_{i=1}^{n-1} (m_i - 1) + \sum_{i=1}^{t} (m_i - 1) + n & n \leq t \end{cases}$$

Now, we determine the star-critical Ramsey number of matchings versus a complete graph. For this purpose, first we characterize the class of all $(K_n, m_1K_2, \dots, m_tK_2)$-free colorings on $r(K_n, m_1K_2, \dots, m_tK_2) - 1$ vertices and then we use such a characterization to find the star-critical Ramsey number $r^*(K_n, m_1K_2, \dots, m_tK_2)$.

Definition 2.2 *Let $m_1, m_2, \dots, m_t$ be given positive integers and $m_1 = max\{m_1, \dots, m_t\}$. Let $t \geq 2$, $n = t + 1$ and $r = r(K_n, m_1K_2, \dots, m_tK_2) = \sum_{i=1}^{t} 2(m_i - 1) + n$. Define the graph $\mathbb{G} \cong K_{r-1}$ with the $(t+1)$-edge coloring with colors $\alpha_0, \alpha_1, \dots, \alpha_t$ as follow:*

$$\mathbb{G}: \mathbb{G}^{\alpha_0} = K_t(2m_1 - 1, 2m_2 - 1, \dots, 2m_t - 1),$$
$$\mathbb{G}^{\alpha_i} = K_{2m_i - 1},$$

for every $i = 1,2, \dots, t$.

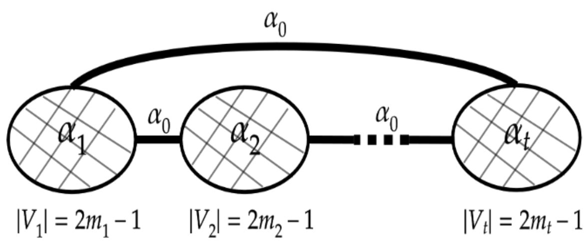

Figure 1: The $(K_n, m_1K_2, \dots, m_tK_2)$-free coloring of $\mathbb{G}$

Theorem 2.3 *Let $t \geq 2$, $n = t + 1$ and $m_1, m_2, \dots, m_t$ be given positive integers with $m_1 = max\{m_1, \dots, m_t\}$. Also, let $r = r(K_n, m_1K_2, \dots, m_tK_2) = \sum_{i=1}^{t} 2(m_i - 1) + n$. If c is a $(K_n, m_1K_2, \dots, m_tK_2)$-free coloring of K_{r-1}, then c is a unique coloring of K_{r-1} described in Definition 2.2.*

Proof. We use induction on m_1 to prove the theorem. Let c be an arbitrary $(K_n, m_1K_2, \dots, m_tK_2)$-free coloring of $G = K_{r-1}$. If $m_1 = 1$, then $m_1 = m_2 = \dots = m_t = 1$ and so, $r = n$. As c is a $(K_n, m_1K_2, \dots, m_tK_2)$-free coloring, then K_{n-1} does not contain any edge of color α_i, $1 \leq i \leq t$. Thus, G is a monochromatic copy of K_{n-1} of color α_0. Therefore, when $m_1 = 1$, the coloring c has same coloring described in Definition 2.2.

$$G: G^{\alpha_0} = K_t(1,1, \dots, 1),$$
$$G^{\alpha_i} = K_1,$$

for every $i = 1,2, \dots, t$. Now, let $m_1 \geq 2$. As $m_1 \geq 2$ and

$$r(K_n, m_1K_2, m_2K_2, \dots, m_tK_2) - 1 \geq r(K_n, m_2K_2 \dots, m_tK_2),$$

then c contain at least an edge of color α_1. Let $e = uv$ be the edge of color α_1. Delete the vertices of e from G and let H be the resulting graph. Clearly

$$V(H) = r(K_n, m_1K_2, \dots, m_tK_2) - 1 - |\{u, v\}| = \sum_{i=1}^{t} 2(m_i - 1) + n - 3.$$

Since $|V(H)| = r(K_n, (m_1 - 1)K_2, m_2K_2, \dots, m_tK_2) - 1$ and the coloring c' induced by c on H is a $(K_n, (m_1 - 1)K_2, m_2K_2, \dots, m_tK_2)$-free coloring, then by the induction hypothesis, c' is $(t+1)$-edge coloring of H as described in Definition 2.2, such that

$$H: H^{\alpha_0} = K_t(2m_1 - 3, 2m_2 - 1, \dots, 2m_t - 1),$$
$$H^{\alpha_i} = K_{2m_i - 1},$$

for every $i = 1,2,\dots,t$. As H^{α_0} is a t-partite graph, let $V_1, V_2 \dots, V_t$ be the partite sets of H such that $|V_1| = 2m_1 - 3$ and $|V_i| = 2m_i - 1$, for $2 \le i \le t$. Now, consider vertices u and v of edge e with the graph H. If there is an edge in $E[\{u,v\}, V(H)]$ of color α_i, for $2 \le i \le t$, then we have a matching of size $m_i K_2$, for some i, which is a contradiction. So, we use colors α_0 and α_1 to color edges $E[\{u,v\}, V(H)]$. Consider the following cases.

Claim 1: $ux \not\in G^{\alpha_0}$, for every $x \in V_1$.

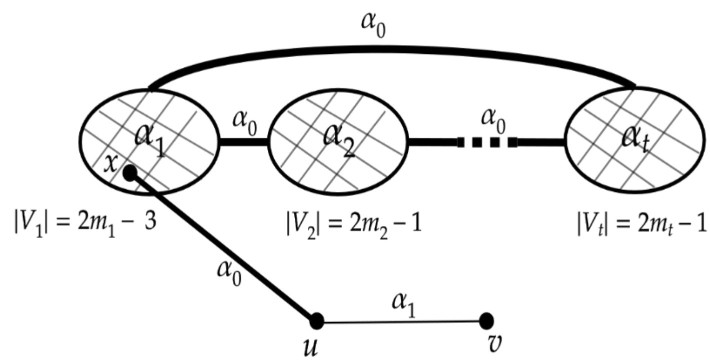

Figure 2: When $ux \in G^{\alpha_0}$, for some $x \in V_1$.

On the contrary, let $ux \in G^{\alpha_0}$, for some $x \in V_1$. Thus, for some i, $2 \le i \le t$, the edges $E[u, V_i]$ do not belong to G^{α_0}, otherwise, u joint with a vertex from each part V_i, $1 \le i \le t$, form a monochromatic copy of K_{t+1} of color α_0, a contradiction. So, without loose of generality, we may assume that $E[u, V_2] \in G^{\alpha_1}$. If there is an edge of color α_1 in $E[v, V_i]$, for $1 \le i \le t$, then it is easy to see that G^{α_1} contains m_1 disjoint copies of K_2, which is impossible. Thus, $E[v, V_i] \in G^{\alpha_0}$, for every $1 \le i \le t$, which form a monochromatic copy of K_{t+1} of color α_0, a contradiction. By a similar argument, $vx \not\in G^{\alpha_0}$, for every $x \in V_1$.

As $E[u, V_1] \in G^{\alpha_1}$ and $E[v, V_1] \in G^{\alpha_1}$, then $E[v, V_i] \in G^{\alpha_0}$, for every $i = 2,\dots,t$, because otherwise, for an edge $vy \in E[v, V_i]$ of color α_1, $G^{\alpha_1}[V_1, u, vy]$ contains $m_1 K_2$, a contradiction. By similar argument, $E[u, V_i] \in G^{\alpha_0}$, for every $i = 2,\dots,t$. Therefore, the resulting coloring of G is as coloring described in Definition 2.2. □

Now, as a corollary of Theorem 2.3, we are ready to prove our first result.

Corollary 2.4 *Let $m_1, \dots, m_t$ be positive integers and $m_1 = max\{m_1, \dots, m_t\}$. For $t \ge 2$ and $n = t + 1$, we have*

$$r^*(K_n, m_1K_2, \dots, m_tK_2) = \sum_{i=1}^{t-1} (2m_i - 1) + 1.$$

Proof. Let $r = r(K_n, m_1K_2, \dots, m_tK_2) = \sum_{i=1}^{t} 2(m_i - 1) + n$ and r^* be the claimed number for $r^*(K_n, m_1K_2, \dots, m_tK_2)$. As $n = t + 1$, then $r = \sum_{i=1}^{t} (2m_i - 1) + 1$. For the lower bound, consider the graph $H = K_{r-1} \sqcup K_{1,r^*-1}$ and partition the vertices of K_{r-1} into t parts $V_1, V_2, \dots, V_t$ such that for every i, $1 \le i \le t$, $|V_i| = 2m_i - 1$. Color all edges contained in V_i, $1 \le i \le t$, by color α_i and all edges between parts by color α_0. Now, add a vertex v adjacent to every vertex in V_i, $1 \le i \le t - 1$, by color α_0. Since $\chi(H^{\alpha_0}) \le t = n - 1$, then H^{α_0} does not contain K_n as a subgraph. Also, for every $1 \le i \le t$, the subgraph

H^{α_i} does not contain matching of size m_i. Therefore, we have a $(K_n, m_1K_2, \dots, m_tK_2)$-free coloring of H and so, $H \nrightarrow (K_n, m_1K_2, \dots, m_tK_2)$.

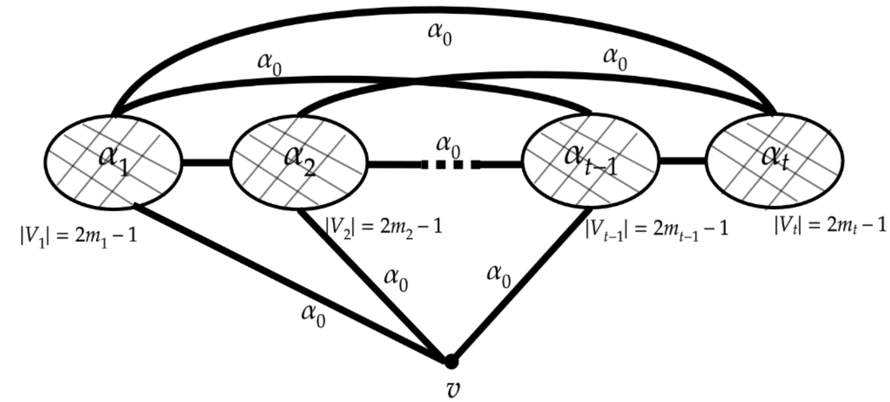

Figure 4: A $(K_n, m_1K_2, \dots, m_tK_2)$-free coloring of $K_{r-1} \sqcup K_{1,r^*-1}$, when $n = t + 1$.

For the upper bound, let $G = K_{r-1} \sqcup K_{1,r^*}$ and let v be the vertex of degree r^* in G. We prove that $G \to (K_n, m_1K_2, \dots, m_tK_2)$. Consider an arbitrary $(t+1)$-edge coloring of G which induced a $(t+1)$-edge coloring of $G \backslash \{v\} \cong K_{r-1}$ and by Theorem 2.3, this coloring is unique as shown in Figure 2.2.

If there is an edge of color α_i, $1 \leq i \leq t$, incident with v, then obviously G contains a matching of size m_i. Therefore, we may assume that all $r^* = \sum_{i=1}^{t-1} (2m_i - 1) + 1$ edges incident with v are color α_0. To avoid a K_{t+1} of color α_0, v can adjacent to at most $(t-1)$ parts $V_{i_1}, \dots, V_{i_{t-1}}$ of the sets $V_1, V_2, \dots, V_t$ in the unique $(t+1)$-edge coloring of K_{r-1}. As $\deg(v) \geq r^*$, it is easy to see that v has at least one neighbour in each V_i, $1 \leq i \leq t$, and hence, G contains a copy of K_{t+1} of color α_0. This means that $G \to (K_n, m_1K_2, \dots, m_tK_2)$. □

For next result, we prove the following theorem and show that for $n \geq t+1$, the star-critical Ramsey number of matchings versus a complete graph is Ramsey-full.

Theorem 2.5 *Let $n \geq t+2$, $n \geq 3$, $m_1, \dots, m_t$ be positive integers and $m_1 = max\{m_1, \dots, m_t\}$, then*

$$r^*(K_n, m_1K_2, \dots, m_tK_2) = \sum_{i=1}^{t} 2(m_i - 1) + n - 1.$$

Proof. Let $r = r(K_n, m_1K_2, \dots, m_tK_2)$ and r^* be the claimed number. Note that

$$r - 1 = \sum_{i=1}^{t} 2(m_i - 1) + n - 1 = \sum_{i=1}^{t} (2m_i - 1) + (n - t - 1).$$

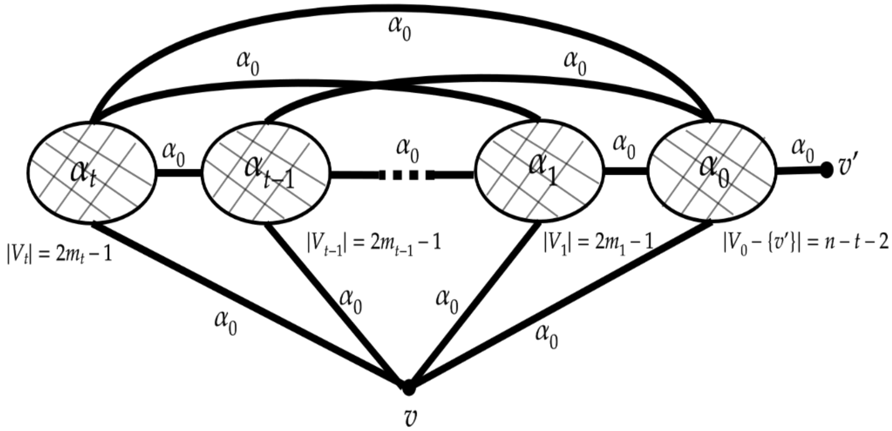

Figure 5: A $(K_n, m_1K_2, \dots, m_tK_2)$-free coloring of $K_{r-1} \sqcup K_{1,r^*-1}$, when $n \geq t+2$.

For the lower bound, we partition the vertices of K_{r-1} into $(t+1)$ sets $V_0, V_1, \dots, V_t$ such that for $i = 1, \dots, t$, $|V_i| = (2m_i - 1)$ and $|V_0| = (n - t - 1)$. Color all edges contained in V_i, $1 \leq i \leq t$, by color α_i and the rest edges by color α_0. Now, add a vertex v adjacent to every vertex in V_i, $1 \leq i \leq t$ and $(n - t - 2)$ vertex from V_0 by color α_0. Since there is a vertex in V_0 which is non-adjacent with v, then the clique number of the H^{α_0} is $(n - t - 2 + t + 1) = n - 1$, and so, H^{α_0} does not contain K_n as a subgraph. For every i, $1 \leq i \leq t$, H^{α_i} does not contain matching of size m_i. Therefore, we have a $(K_n, m_1K_2, \dots, m_tK_2)$-free coloring of H and so $H \nrightarrow (K_n, m_1K_2, \dots, m_tK_2)$.

As $r^*(K_n, m_1K_2, \dots, m_tK_2) \leq r(K_n, m_1K_2, \dots, m_tK_2) - 1$, then $\sum_{i=1}^{t} 2(m_i - 1) + n - 1 = r(K_n, m_1K_2, \dots, m_tK_2) - 1$ is an upper bound for $r^*(K_n, m_1K_2, \dots, m_tK_2)$ and the proof is complete. □

Now, proof of the main theorem (Theorem1.1) is an immediate consequence of Corollary 2.4 and Theorem 2.5.

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