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Rainbow domination of graphs

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ABSTRACT

Let G=(V,E) be a graph with the vertex set V=V(G) and the edge set E=E(G). Let k be a positive integer, [k]={1,2,…,k} and $\mathcal{P}$([k]) be the power set of [k]. A function f:V(G)→$\mathcal{P}$([k]) is a k-rainbow dominating function if for every vertex x with f(x)=ϕ, f(N(x))=[k]. The k-rainbow domination number γ_{rk}(G) is the minimum weight of $\sum_{x\in V(G)}|f(x)|$ taken over all k-rainbow functions. We investigate the rainbow domination and independent rainbow domination numbers of classes of graphs.

KEYWORDS: Rainbow domination, independent rainbow domination, graphs.

1 INTRODUCTION

Let G=(V,E) be a simple graph with the vertex set V=V(G) and the edge set E=E(G). The order of G is the number of vertices of G. For any vertex v∈V, the open neighborhood of v is the set N(v)={u∈V| uv∈E} and the closed neighborhood of v is the set N[v]=N(V)∪{v}. For a set S⊆V, the open neighborhood of S is N(S)=$\bigcup_{v\in S} N(v)$ and the closed neighborhood of S is N[S]=N(S)∪S.

The Cartesian product G□H of graphs G and H is a graph such that the vertex set of G□H is the Cartesian product V(G)×V(H) and any two vertices (u,u') and (v,v') are adjacent in G□H if and only if either u=v and u' is adjacent to v' in H, or u'=v' and u is adjacent to v in G.

A set S⊆V is a dominating set if N[S]=V, or equivalently, every vertex in V-S is adjacent to at least one vertex in S. The domination number γ(G) is the minimum cardinality of a dominating set in G. A dominating set with cardinality γ(G) is called a γ(G)-set. The family of all γ(G)-sets of a graph G is denoted by Γ(G).

In 2008, Bresar et al. [3] have introduced the k-rainbow domination as a generalization of domination in graphs.

Definition 1. ([3]) Let k be a positive integer,[k]={1,2,…,k} and $\mathcal{P}$([k]) be the power set of [k]. For any graph G, a function f:V(G)→$\mathcal{P}$([k]) is a k-rainbow dominating function (or simply kRDF) if for every vertex v∈V with f(v)=ϕ, f(N(v)) =$\bigcup_{u\in N(V)} f(u) = [k]$. The weight w(f) of a kRDF f is defined as w(f)=$\sum_{v\in V(G)}|f(x)|$. The minimum weight of a kRDF of G is called the k-rainbow domination number of G, and is denoted by γ_{rk}(G).

A kRDF f is an independent k-rainbow dominating function (IkRDF) if no two vertices assigned nonempty sets are adjacent. The weight of an IkRDF f is the value $w(f)=\sum_{v\in V(G)} |f(x)|$.
The independent k-rainbow domination number $\gamma_{irk}(G)$ is the minimum weight of an IkRDF on G.
One reason for the popularity of domination problems is the wide range of applicability and directions of possible research. One of the results may by used to the proof of results in this manuscript has been showed by Bresar et al. [3]. They showed that for any graph G and complete graph K_k, $\gamma_{rk}(G) =\gamma(G\square K_k)$.
This observation together with Vising's conjecture stimulated the search for graphs for which $\gamma=\gamma_{r2}$ (see also [6]). Note that, the Vising's upper bound $\gamma_{rk}(G)\leq k\gamma(G)$ [10].
Chang et al. [5] were quick on the uptake and showed that, for positive integer k, the k-rainbow domination problem is NP-complete, even when restricted to chordal graphs or bipartite graphs. The paper also shows that the problem remains NP-complete on planar graphs.
Notice that the above discussion shows that $\gamma_{rk}(G)$ is a non-decreasing function in k. Chang et al. [5] showed that for all graphs G on n vertices and all positive integer k, $\min\{k,n\}\leq\gamma_{rk}(G)\leq n$.
Algorithm: Among all vertices of a graph G we choose a vertex u of the maximum degree and assign $\{1,2\}$ to it and ϕ to its neighbors. Consider the subgraph $G_u=G-N[u]$. Iterate this process until $\Delta(G_u)\leq 2$. Therefore, G_u would be a disjoint union of path and cycles. We now use the formulas of the 2-ainbow domination number for paths and cycles.

2 RAINBOW DOMINATION AND INDEPENDENT RAINBOW DOMINATION OF GRAPHS

The rainbow domination number of very few families of graphs are known exactly.
In this section, we study the rainbow domination number of some families of graphs.

2.1 Hypercube Q_n

The Hypercube or (n-cube) Q_n is a graph whose vertex set is the set of all n-dimensional boolean vectors, two vertices being joined if and only if they differ in exactly one coordinate. The determination of the domination parameters of Q_n is a significant unsolved problem.
Bresar et al. [3] proved that $\gamma_{rk}(G)=\gamma(G\square K_k)$ for any graph G. In particular, since $Q_{n+1}=Q_n\square K_2$, we infer that

$$\gamma_{r2}(Q_n)=\gamma(Q_{n+1}), \qquad \text{for } n \geq 1 \qquad (1)$$

Arumugam et al. [2] showed that $\gamma(Q_5)=7$, $\gamma(Q_6)=12$, $\gamma(Q_7)=16$ and $\gamma(Q_n)\leq 2^{n-3}$for all $n\geq 7$. Moreover, they proved:
Corollary 2. ([2] Theorem 2.6) If $n=2^k-1$, then $\gamma(Q_n)=2^{n-k}$.
Arumugam et al. [2] also showed $\gamma(Q_n)\leq 2^{n-3}$for all $n\geq 7$. We can then improve this result as follows.
Proposition 3. Let $k\geq 4$ be an integer and $2^{k-1}-1\leq n\leq 2^k-2$. Then $\gamma(Q_n)\leq 2^{n-k+1}$.
Proof. If $m=2^{k-1}-1$ then $\gamma(Q_m)=2^{n-k+1}$. If $n=m+1=2^{k-1}$, then $\gamma(Q_n)\leq 2\gamma(Q_m) =2^{m-k+2}=2^{n-k+1}$. Now if $n=m+r$ and $0\leq r\leq 2^{k-1}-1$, then $\gamma(Q_n)\leq 2^r\gamma Q_m)=2^{m-k+1+r}=2^{n-k+1}$.□
We observed that $\gamma(Q_n)=2^{n-k}$ for $n=2^k-1$ and $k\geq 3$. Proposition 5 shows that if $p=n-1=2^k-2$, then $\gamma(Q_p)\leq 2^{p-k+1} =2^{n-k}$. But we feel that we can infer $\gamma(Q_{n-1})<2^{n-k}$. The sentence
$\gamma(Q_n) =2^{n-k}$ for $n=2^k-1$ implies that each vertex of dominating set should dominate n vertices other than itself and any two of them do not dominate common vertex and 2^{n-k-1} of number of dominating vertices should be chosen from every Q_{n-1} of Q_n. Now let $p=n-1=2^k-2$. Then

$2^{n-k-1}=2^{p-k}$ of Q_p dominate $2^{p-k}p+2^{p-k}=2^p-2^{p-k}$. The 2^{p-k} vertices need 2^{p-k-1} vertices for dominated. Therefor $\gamma(Q_p)\leq 2^{p-k}+2^{p-k-1}=3(2^{p-k-1})<2^{p+1-k}=2^{n-k}$. We also have the following result from [7].

Theorem 4. ([7], Theorem 3.1) If $m=2^k$, then $\gamma(Q_m)=2^{m-k}$.

Now, as an immediate consequence of (1) and given results, we establish the following result that its proof is straightforward and it is left.

Proposition 5. For 2-rainbow domination of hypercube Q_n, we have.

(i) $\gamma_{r2}(Q_m)=\gamma(Q_{m+1})$, for $0\leq m\leq 3$.

(ii) $\gamma_{r2}(Q_4)=7$.

(iii) $\gamma_{r2}(Q_5)=12$.

(iv) $\gamma_{r2}(Q_6)=16$.

(v) $\gamma_{r2}(Q_7)=32$.

(vi) $\gamma_{r2}(Q_n)=2^{n+1-k}$, for $2^k-2\leq n\leq 2^k-1$.

(vii) $\gamma_{r2}(Q_n)=\gamma(Q_{n+1})\leq 2^{n+1-k}$, for $2^k\leq n\leq 2^{k+1}-4$.

(viii) $\gamma_{r2}(Q_n)=\gamma(Q_{n+1})\leq 3(2^{n-1-k})$, for $n=2^{k+1}-3$.

2.2 Harary graphs

The domination parameters of Harary graphs have been studied in [8]. Here we establish the 2-rainbow domination of Harary graphs. Let us recall the Harary graphs. Given positive integers $k < n$, place n vertices around a circle, equally spaced. If k is even, form $H_{k,n}$ by making each vertex adjacent to the nearest k/2 vertices in each direction around the circle. If k is odd and n is even, form $H_{k,n}$ by making each vertex adjacent to the nearest (k-1)/2 vertices in each direction and to the diametrically opposite vertex. $H_{k,n}$. In each case, $H_{k,n}$ is k-regular.

When k and n are both odd, index the vertices by the integers modulo n. Construct $H_{k,n}$, from $H_{k-1,n}$ by adding the edges $i\longleftrightarrow i+(n-1)/2$ for $0\leq i\leq (n-1)/2$.

Lemma 6. Let $H_{k,n}$ $(2\leq k<n)$ be a Harary graph.

(i) If k is an even integer and $n=q(k+2)+r$, $0\leq r\leq k+1$, then

$$\gamma_{r2}(H_{k,n})=\begin{cases} 2q & if\ r=0 \\ 2q+1 & if\ r=1 \\ 2(q+1) & if\ 2\leq r\leq k+1 \end{cases}$$

(ii) If k is an odd integer and $n=q(k+1)+r$, $0\leq r\leq k$, then

$$\gamma_{r2}(H_{k,n})=\begin{cases} 2q & if\ r=0 \\ 2q+1 & if\ r=1 \\ 2(q+1) & if\ 2\leq r\leq k+1 \end{cases}$$

Proof. (i) Let k be even. First we show that for any k+2 consecutive vertices $v_{i1},v_{i2},\ldots,v_{i(k/2)+1},v_{i(k/2)+2},\ldots,v_{ik+1},v_{ik+2}$, there exist at least two vertices v_{il} and v_{it} with $f(v_{il})\cup f(v_{it})=\{1,2\}$ or there exists one vertex v_{il} with $f(v_{il})=\{1,2\}$. Suppose on the contrary, if there exists only one vertex like v_{il} with value {1} or {2}, then we cannot assign any value to $v_{il+(k/2)+1}$ or $v_{il-(k/2)-1}$, a contradiction.

Now we give a 2-rainbow dominating function as follows:

Let $n=q(k+2)$ and $V(H_{k,n})=\{v_1,\ldots,v_{(k/2)+2},\ldots,v_{k+3},\ldots,v_{q(k+2)}\}$. We define $f:V(H_{k,n})\rightarrow\mathcal{P}(\{1;2\})$ by

$$f(v_{m((k/2)+1)+1})=\begin{cases} \{1\} & if\ 0\leq m\leq 2q-1 \text{ is even} \\ \{2\} & if\ 0\leq m\leq 2q-1 \text{ is odd} \\ \phi & otherwise \end{cases}$$

Let n=q(k+2)+1 and $V(H_{k,n})=\{v_1,\dots,v_{(k/2)+2},\dots,v_{k+3},\dots,v_{q(k+2)},v_{q(k+2)+1}\}$.
We define $f : V(H_{k,n}) \to \mathcal{P}(\{1;2\})$ by

$$f(v_{m((k/2+1)+1)})=\begin{cases}\{1\} & if\ 0 \le m \le 2q \text{ is even}\\ \{2\} & if\ 0 \le m \le 2q-1 \text{ is odd}\\ \phi & otherwise\end{cases}$$

Let n=q(k+2)+r where $2 \le r \le k+1$ and
$V(H_{k,n})=\{v_1,\dots,v_{(k/2)+2},\dots, v_{k+3},\dots, v_{q(k+2)}; v_{q(k+2)+1},\dots,v_{q(k+2)+r}\}$.

We define $f : V(H_{k,n}) \to \mathcal{P}(\{1,2\})$ by $f(vq_{(k+2)+1+\lfloor r/2 \rfloor})=\{2\}$,

$$f(v_{m((k/2)+1)+1})==\begin{cases}\{1\} & if\ 0 \le m \le 2q \text{ is even}\\ \{2\} & if\ 0 \le m \le 2q-1 \text{ is odd}\end{cases}$$

and $f(v_i)=\phi$ for otherwise. Therefore we have the desired.
(ii) Let k be odd. We can use a method similar to that of (i) to establish the result.□

2.3 Complete and complete r(≥2)-partite graphs

In what fallows the kRDF and IkRDF of complete graphs and complete r(≥2)-partite graphs are verified via observations which have simple proofs.

Observation 7. For any positive integer n,

$$\gamma_{rk}(K_n)=\begin{cases}k & if\ n \ge k,\\ n & if\ n < k.\end{cases}$$

$$\gamma_{irk}(K_n)=\begin{cases}1 & if\ n = 1\\ k & if\ n \ge 2\end{cases}$$

Observation 8. Let $G=K_{m1,m2,\dots,mr}$ be a complete r(≥2)-partite graph with $m1 \le m2 \le \dots \le mr$. Then,

$$\text{(i)}\gamma_{rk}(G)=\begin{cases}\sum_{i=1}^{i=r} mi & \text{if } \sum_{i=1}^{i=r} mi \le k,\\ k & if\ 1 \le m1 \le k\ \ and\ \sum_{i=1}^{i=r} mi \ge k,\\ m1 & if\ k+1 \le m1 \le 2k-1,\\ 2k & if\ m1 \ge k+1\end{cases}$$

$$\text{(ii) } \gamma_{irk}(G)=\begin{cases}k & if\ 1 \le m1 \le k\\ m1 & if\ m1 \ge k+1\end{cases}$$

2.4 Paths, cycles and wheels

The k-rainbow domination numbers of the paths, cycles and generalized Petersen graphs have already been considered elsewhere. In this section, we study the independent 3-rainbow domination number of paths and the independent 2,3-rainbow domination numbers of cycles and provide a construction for I3RDF with the desired weight in each case.
The independent 2-rainbow domination number of trees has been studied in [1]. For paths we have the following.

Observation 9. ([1]) $\gamma_{ir2}(P_n)= \lfloor n/2 \rfloor +1$.

Proposition 10.

$$\gamma_{ir3}(P_n)=\begin{cases}\lceil \frac{3n+1}{4} \rceil & if\ n = 0\ or\ 1\ (mod\ 4)\\ \lceil \frac{3n+3}{4} \rceil & if\ n = 2\ or\ 3\ (mod\ 4)\end{cases}$$

Proof. Let $v_1,\dots,v_n$ be the vertices of the path P_n. First of all, without loss of generality, by $|f(v_i)|=0$, $|f(v_i)|=1$, $|f(vj)|=2$ and $|f(v_k)|=3$ we mean that $f(v_i)=\phi$, $f(v_i)=\{1\}$, $f(v_j)=\{2,3\}$ and $f(v_k)=\{1,2,3\}$, respectively. It is well known that if for a vertex v_i, $f(v_i)=\phi$, then for the independent 3-rainbow domination one of the following must be held:
(i) $|f(v_{i-1})|=3$ or $|f(v_{i+1})|=3$, (ii) $|f(v_{i-1})|=1$ and $|f(v_{i+1})|=2$, or (iii)$|f(v_{i-1})|=2$ and $|f(v_{i+1})|=1$.

We now prove the result by induction on n. For n∈{4,5,6,7} it is easy to see that $\gamma_{ir3}(P_4)=4 = \lceil(3n+1)/4\rceil$ by assigning $|f(v_1)|=1$, $|f(v_3)|=3$ and $|f(v_i)|=0$ for i=2,4, $\gamma_{ir3}(P_5)=4=\lceil(3n+1)/4\rceil$ by assigning $|f(v_1)|=1=|f(v_5)|$, $|f(v_3)|=2$ and $|f(v_i)|=0$ for i=2, 4,$\gamma_{ir3}(P_6)=6=\lceil(3n+3)/4\rceil$ by assigning $|f(v_1)|=1$, $|f(v_3)|=2$, $|f(v_5)|=3$ and $|f(v_i)|=0$ for i=2,4, 6,$\gamma_{ir3}(P_7)=6=\lceil(3n+3)/4\rceil$ by assigning $|f(v_1)|=1$, $|f(v_3)|=2$, $|f(v_5)|=1$, $|f(v_6)|=2$ and $|f(v_i)|=0$ for i=2,4,7. Hence the basis of induction holds. Let n=4t, t≥2 and the result holds for n=4t-4 by assigning $|f(v_i)|=1$ for i≡1(mod 4), $|f(v_i)|=2$ for i≡3 (mod 4) and i≠4t-5, $|f(v_{4t-5})|=3$ and $|f(v_i)|=0$ for otherwise. Hence $\gamma_{ir3}(P_{4t-4})=\lceil((3(4t-4)+1))/4\rceil$. For n=4t, t≥2, we assign $|f(v_i)|=1$ for i≡1 (mod 4), $|f(v_i)|=2$ for i≡3 (mod 4) and i≠4t - 1, $|f(v_{4t-1})|=3$ and $|f(v_i)|=0$ for the other vertices v_i. Therefore
$\gamma_{ir3}(P_{4t})=\gamma_{ir3}(P_{4t-4})+3=\lceil(3(4t-4)+1)/4\rceil+3=\lceil(3(4t-4)+1+12)/4\rceil=\lceil(3n+1)/4\rceil$.

For n≡1,2,3 (mod 4), the proofs are similar.□

We have the exact formulas for $\gamma_{ir2}(P_n)$ and $\gamma_{ir3}(P_n)$. In what follows we give the exact formula for $\gamma_{irk}(P_n)$, for k≥4. We have $\gamma_{irk}(P_1)=1$, $\gamma_{irk}(P_2)=\gamma_{irk}(P_3)=k$.

So, we may assume that n≥4.

Proposition 11. For n,k≥4, $\gamma_{irk}(P_n)=\begin{cases} kt+1 & if\ n=4t\ or\ 4t+1 \\ k(t+1) & otherwise \end{cases}$

Proof. Let $f:V(P_n)\rightarrow\mathcal{P}(\{1, 2, \ldots,k\})$ be a $\gamma_{irk}(P_n)$-function. Let P_n be a path with vertices $v_1,\ldots,v_n$. We consider four cases.

Case 1. n=4t. It is easy to see that $w(f(P_4))=k+1$. Consider the subpaths $P_i:v_{4i-3}v_{4i-2}v_{4i-1}v_{4i}$, for 1≤i≤t. It is straightforward to see that $w(f|_{P_i})\geq k$, for all 1≤i≤t. Therefore, $\gamma_{irk}(P_n)=w(f)\geq kt$. Suppose now that w(f)=kt. Therefore, $w(f|_{P_i})=k$, for all 1≤i≤t. Now if three vertices of a subpath P_i have the weight ϕ under f, then $w(f|_{P_{i-1}})>k$ or $w(f|_{P_{i+1}})>k$. This is a contradiction.

So, exactly two vertices of P_i have the weight ∅ under f. We now show that vertex $|f(v_{4i})|=0$ for 1≤i≤t-1 where t≥2. It is clear that $|f(v_4)|=0$. Suppose that $|f(v_{4i})|=0$ for 1≤i≤t-2. If $|f(v_{4i+1})|=0$, then $f(v_{4i+2})|=k$ and $|f(v_{4(i+1)})|\geq1$, is a contradiction. Thus $|f(v_{4i+1})|=[j]$, $|f(v_{4i+2})|=0$, $|f(v_{4i+3})|=[k] \setminus [j]$ and $|f(v_{4(i+1)})|=0$. Now we clime that $|f(\bigcup_{i=4t-3}^{4t}\{vi\})|\geq k+1$. Because if $|f(v_{4t-3})|=0$, then $|f(v_{4t-2})|=k$ and $|f(v_{4t-1})\cup f(v_{4t})|\geq1$, and if $|f(v_{4t-3})|\geq1$, then $|f(v_{4t-2})\cup f(v_{4t-1})\cup f(v_{4t})|\geq k$. Therefore $\gamma_{irk}(P_n)=w(f)\geq kt+1$ for n=4t.

On the other hand, we give an independent k rainbow dominating function g with w(g) = kt + 1. Let $g:V(P_n)\rightarrow\mathcal{P}(\{1, 2, \ldots, k\})$ defined by

$$g(v)=\begin{cases} \phi & v = v\,4i-2, v4i(1 \leq i \leq t) \\ \{1\} & v=v4i-3,(1 \leq i \leq t) \\ \{2,\ldots,K\} & v=v4i-1(1 \leq i \leq t-1), \\ \{1,\ldots,K\} & v=v4t-1 \end{cases}$$

is an IkRDF of P_n with weight kt+1. So, $\gamma_{irk}(P_n)\leq kt+1$. This shows that $\gamma_{irk}(P_n)=kt+1$.

Case 2. n=4t+1. Similar to Case 1 we can show that w(f)≥kt+1.

Now the function $g':V(P_n)\rightarrow\mathcal{P}(\{1,2, \ldots,k\})$ is defined by

$$g'(v)=\begin{cases} \{1\} & v=v4i-3,(1 \leq i \leq t) \\ \phi & v=v4i-2, v4i\ (1 \leq i \leq t) \\ \{2,\ldots,k\} & v=v4i-1(1 \leq i \leq t) \end{cases}$$

is an IkRDF of P_n with weight kt+1. This shows that $\gamma_{irk}(P_n)=kt+1$.

Case 3. n=4t+2. Let first n=6. By assigning [1] to v_1, [k]\[1] to v_3, [k] to v_5 and ϕ to the vertices v_2, v_4, v_6, we have the exact value $\gamma_{irk}(P_6)=2k$. Suppose now that n=4t+2 where

$t \geq 2$. $P_{4t} = P_n - \{v_{4t+1}, v_{4t+2}\}$, and f is an IkRDF of P_n. Using the Case 1, it has been seen $w(f|_{P_{4t}}) \geq kt$ provided that $f(v_{4t-1})=[k-1]$. In this case we should assign $f(v_{4t+1})=[k]$ in P_n. Thus $\gamma_{irk}(P_n) \geq k(t+1)$.

Now the function $h: V(P_n) \to \mathcal{P}(\{1,2,\ldots,k\})$ is defined by

$$h(v)=\begin{cases}\{1\} & v = v4i-3, v4i(1 \leq i \leq t)\\ \phi & v = v4i-2, (1 \leq i \leq t+1), \text{and } v = v4i(1 \leq i \leq t)\\ \{2,\ldots,K\} & v = v4i-1(1 \leq i \leq t-1),\\ \{1,2,\ldots,K\} & v = v4t+1\end{cases}$$

an IkRDF of P_n with weight $k(t+1)$. This shows that $\gamma_{irk}(P_n)=k(t+1)$.

Case 4. $n=4t+3$. We have $w(f|_{P_i})) \geq k$, for all $1 \leq i \leq t$. So by Case 3, $\gamma_{irk}(P_n)=w(f) \geq k(t+1)$.

Now the function $h': V(P_n) \to \mathcal{P}(\{1,2,\ldots,k\})$ is defined by

$$h'(v)=\begin{cases}\{1\} & v = v4i-3,\\ \phi & v = vm, \text{for even positive integer } m\\ \{2,\ldots,k\} & v = v4i-1(1 \leq i \leq t+1)\end{cases}$$

It is easy to see that h' is an IkRDF of P_n with weight $k(t+1)$. So, $\gamma_{irk}(P_n)=k(t+1)$. This complete the proof.□

The 2-rainbow domination of a cycle has been studied in [4].

Observation 12. ([4], Proposition 3.2) For $n \geq 3$, $\gamma_{r2}(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$

The independent k-rainbow domination of a cycle is studied here and we known that $\gamma_{r2}(G) \leq \gamma_{ir2}(G)$. For cycle C_3 it is easy to see $\gamma_{ir2}(C3)=2$. For $\gamma_{ir2}(C_n)$ we have the following.

Proposition 13. For $n \geq 4$, $\gamma_{ir2}(C_n)=\begin{cases}\frac{n}{2} & if\ n = 4k\\ \lceil \frac{n}{2} \rceil + 1 & otherwise\end{cases}$

Proof. For any cycle C_n, Observation 12 implies that $\gamma_{ir2}(C_n) \geq \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$.

Let $n \equiv 0 \pmod 4$. By assigning value 1 to v_{4i+1}, value 2 to v_{4i+3} for $0 \leq i \leq \frac{n}{4}-1$ and ϕ to the others. Then $\gamma_{ir2}(C_n)=\frac{n}{2}$ for $n \equiv 0 \pmod 4$.

Let $n \equiv 2 \pmod 4$. By assigning value 12 to v_1, value 1 to v_{4i+1}, for $1 \leq i \leq \frac{n-2}{4}$, value 2 to v_{4i+3} for $0 \leq i \leq \frac{n-6}{4}$ and ϕ to the others. Then $\gamma_{ir2}(C_n) = \lfloor \frac{n}{2} \rfloor + 1$ for $n \equiv 2 \pmod 4$.

Let n be odd and let f be a 2-rainbow dominating function of C_n of minimum weight. There is a vertex $x \in V(C_n)$ with $f(x)=\{1,2\}$. Then we get $w(f) \geq 2+\gamma_{ir2}(P_{n-3})=2+\lfloor \frac{n-3}{2} \rfloor+1=2+\frac{n-3}{2}+1 = \frac{n+1}{2}+1= \lceil \frac{n}{2} \rceil + 1.$□

Independent 3-rainbow domination number of cycle C_3 is 3. In the follow we establish the independent 3-rainbow domination number of any cycle with order $n \geq 4$.

Proposition 14. For $n \geq 4$, $\gamma_{ir3}(C_n)=\begin{cases}3\lceil \frac{n}{4} \rceil + 1 & if\ n = 4k+3\\ 3\lceil \frac{n}{4} \rceil & otherwise.\end{cases}$

Proof. Let $n \equiv 0 \pmod 4$. Then it is clear that $\gamma_{ir3}(C_n) \leq \frac{3n}{4}$. Let f be a 3-rainbow dominating function with minimum weight. There is a vertex v_i with $f(v_i)=0$. Let $P_{n-1}=C_n-\{v_i\}$. Then $\gamma_{ir3}(C_n) \geq \gamma_{ir3}(P_{n-1}) = \lceil \frac{3(n-1)+3}{4} \rceil = \frac{3n}{4}$ by Proposition 10. Therefore $\gamma_{ir3}(C_n) = \frac{3n}{4}$ for $n \equiv 0 \pmod 4$.

Let n≡1 (mod 4). Then $\gamma_{ir3}(C_n) \leq 3\lceil \frac{n}{4} \rceil$. Let f be a 3-rainbow dominating function with minimum weight on C_n. There is a vertex vi with $f(v_i)=3$. Then we get $w(f) \geq 3+\gamma_{ir3}(P_{n3})=3+\lceil \frac{3n-6}{4} \rceil = 3+\lceil \frac{3n+3}{4} \rceil = 3\lceil \frac{n}{4} \rceil$, by proposition 10. Therefore, the result holds. Two other parts can be proved similarly.□

We have already obtained the exact value for $\gamma_{ir2}(C_n)$, $\gamma_{ir3}(C_n)$ and $\gamma_{irk}(P_n)$. In what follows we give the exact value for $\gamma_{irk}(C_n)$, for k≥4. We have easily seen that $_{irk}(C_3)=k=\gamma_{irk}(C_4)$ and $\gamma_{irk}(C_5)=2k=\gamma_{irk}(C_6)$. In general we have the following.

Proposition 15. For n≥7 and k≥7, $\gamma_{irk}(C_n)=\begin{cases} kt & if\ n=4t \\ k(t+1) & if\ n=4t+2\ or\ n=4t+1 \\ k(t+1)+1 & if\ n=4t+3 \end{cases}$

Proof. Let n=4t. Since at least 2t vertices should be assigned by ϕ and any such vertex must be adjacent of vertices with weight at least [k], then any IkRDF f with $f(v_{4i-3})=1$, $f(v_{4i-1})=\{2,\ldots,k\}$ and $f(v_{4i-2})=f(v_{4i})=\phi$ for 1≤i≤t is a γ_{irk}-function for C_{4t}. Therefore $\gamma_{irk}(C_{4t})=kt$.

Let n=4t+2. By deleting three vertices v_n, v_{n-1}, v_{n-2} from C_n we will have a path P_{4t-1}. By Proposition 11 $\gamma_{irk}(P_{4t-1})=kt$. Two vertices v_n and v_{n-2} of these three vertices should be assigned by ϕ and the vertex v_{n-1} by [k]. Therefore $\gamma_{irk}(P_{4t+2})=k(t+1)$. Let n≥7 is an odd integer. Then for any IkRDF f, there are two consecutive vertices which will be assigned ϕ by f. If these two vertices are v_i and v_{i+1}, then without lose of generality we should assign $f(v_{i-1})=f(v_{i+2})=[k]$ and $f(v_{i-2})=f(v_{i+3})=\phi$. Thus these six vertices have weight 2k. Now let n=4t +1. Then deleting these consecutive six vertices from C_n we obtain a path $P_{4(t-2)+3}$. By Proposition 11, $\gamma_{irk}(P_{4(t-2)+3})=k(t-1)$. Therefore $\gamma_{irk}(C_{4t+1})=k(t+1)$.

Let n=4t+3. Then deleting these consecutive six vertices from C_n we obtain a path $P_{4(t-1)+1}$. By Proposition 11, $\gamma_{irk}(P_{4(t-1)+1})=k(t-1)+1$. Therefore $\gamma_{irk}(C_{4t+3})=k(t+1)+1$.□

It is easy to see that:

Corollary 16. Let W_n, n≥3, be the wheel graph. Then $\gamma_{irk}(W_n) = k$.

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