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Solving fractional optimal control of systems described by the fractional order differential equations by using Bernoulli wavelets

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ABSTRACT

This paper presents a new numerical method for a class of fractional optimal control problems (FOCPs). The fractional derivative is described in the Caputo sense. The performance index of a FOCP is considered as a function of both the state and the control variables, and the dynamic constraints are expressed by a set of fractional differential equations (FDEs). The method is based upon Bernoulli wavelets. The Bernoulli wavelets is first introduced. The operational matrices of fractional Riemann-Liouville integration and multiplication are derived and are utilized to reduce the given optimization problem to system of algebraic equations. Numerical solutions are presented to demonstrate the feasibility of the method.

KEYWORDS: Optimal control; Fractional calculus; Bernoulli wavelets; Operational matrix; Numerical solution

1 INTRODUCTION

Optimal control problems (OCPs) appear in engineering, science, economics, and many other fields. An extensive body of work exists in the area of optimal control of integer order dynamic systems (Hestenes, 1966; Bryson and Ho, 1975; Gregory and Lin, 1992). It was shown recently that fractional derivatives provide more accurate behavior of a dynamic system (Podlubny, 1999). Therefore, formulations and numerical schemes for OCPs which account for fractional dynamics of these systems would be necessary. Agrawal defines a fractional dynamic system (FDS) as a system whose dynamics is described by FDEs, and a FOCP as an optimal control problem for a FDS (Agrawal, 2004). In Agrawal (2004) and Agrawal and Baleanu (2007) the authors have proposed a general formulation and solution scheme for FOCPs. There also exist other numerical simulations for FOCPs with Riemann-Liouville fractional derivatives such as Tricaud and Chen (2010). In Agrawal (2008a), the necessary conditions of optimization are achieved for FOCPs with the Caputo fractional derivative. There exist numerical simulations for such problems such as Agrawal (2008a), Agrawal (2008b) and Lotfi et al. (2011), where the authors have solved the problem by solving the necessary conditions approximately. In Agrawal (2008b), the problem is solved by a discrete iterative method and in Lotfi et al. (2011) the authors have solved the problem by Legendre orthonormal basis and the operational matrix of fractional integration.

[1] Speaker

The motivation of this paper is to use of the Bernoulli wavelet operational matrix of the fractional integration to solve fractional optimal control problem, by reducing it to the solution of algebraic equations.
The paper is organized as follows: in Section 2, we introduce some necessary definitions of fractional calculus and the Bernoulli wavelets. In Section 3 and 4, we derive the Bernoulli wavelet operational matrix of the fractional integration and the operational matrix of multiplication, respectively. Section 5 is devoted to the problem statement. In Section 6, we report our numerical findings and demonstrate the accuracy of the proposed numerical scheme and concluding remarks are given in the last section.

2 PRELIMINARIES AND NOTATIONS

2.1 Fractional derivative and integral

In this section, we first state some definitions and basic properties regarding fractional derivative and integral.

Definition 1. The Riemann-Liouville fractional integral operator of order q is defined as (Podlubny, 1999)

$$(I^q f)(t) = \begin{cases} \frac{1}{\Gamma(q)} \int_0^t \frac{f(s)}{(t-s)^{1-q}} ds, & q > 0, \\ f(t), & q = 0. \end{cases} \tag{1}$$

Definition 2. Caputo's fractional derivative of order q is defined as

$$(D^q f)(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds, \qquad n-1 < q \le n, n \in \mathrm{N},$$

where $q > 0$ is the order of the derivative and n is the smallest integer greater than q.
For the Riemann-Liouville integral and Caputo derivative we have (Diethelm et al., 2005)

$$D^q I^q f(t) = f(t),$$
$$I^q D^q f(t) = f(t) - \sum_{i=0}^{n-1} f^{(i)}(0) \frac{t^i}{i!}. \tag{2}$$

2.2 Properties of the Bernoulli wavelets

Bernoulli wavelets $\psi_{n,m}(t) = \psi(k, \hat{n}, m, t)$ have four arguments; $\hat{n} = n-1$, $n = 1,2,3,\ldots,2^{k-1}$, k can assume any positive integer, m is the order for Bernoulli polynomials and t is the normalized time. They have defined on the interval $[0,1)$ as follows (Keshavarz et al., 2014)

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k-1}{2}} \tilde{\beta}_m(2^{k-1}t - \hat{n}), & \frac{\hat{n}}{2^{k-1}} \le t < \frac{\hat{n}+1}{2^{k-1}}, \\ 0, & otherwise \end{cases}$$

with

$$\tilde{\beta}_m(t) = \begin{cases} 1, & m = 0, \\ \frac{1}{\sqrt{\frac{(-1)^{m-1}(m!)^2}{(2m)!}\alpha_{2m}}} \beta_m(t), & m > 0, \end{cases}$$

where $m = 0,1,2,\ldots,M$ and $n = 1,2,\ldots,2^{k-1}$. Here, $\beta_m(t)$ are the well-known Bernoulli polynomials of order m which can be defined by (Costabile et al., 2006)

$$\beta_m(t) = \sum_{i=0}^{m} \binom{m}{i} \alpha_{m-i} t^i,$$

where $\alpha_i, i = 0,1,\ldots,m$ are Bernoulli numbers and be defined by the identity

$$\frac{t}{e^t - 1} = \sum_{i=0}^{\infty} \alpha_i \frac{t^i}{i!}.$$

The first few Bernoulli numbers are

$$\alpha_0 = 1,\ \alpha_1 = \frac{-1}{2},\ \alpha_2 = \frac{1}{6},\ \alpha_4 = \frac{-1}{30},\ldots$$

with $\alpha_{2i+1} = 0, \quad i = 1,2,3,\ldots$.

The first few Bernoulli polynomials are

$$\beta_0(t) = 1, \quad \beta_1(t) = t - \frac{1}{2}, \quad \beta_2(t) = t^2 - t + \frac{1}{6}, \quad \beta_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t,\ldots$$

According to Kreyszig (1978), Bernoulli polynomials form a complete basis over the interval $[0,1]$.

2.3 Function approximation

A function $f(t)$ defined over $[0,1)$ may be expressed in terms of Bernoulli wavelets as

$$f(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M} c_{nm} \psi_{nm}(t) = C^T \Psi(t) \tag{3}$$

where T indicates transposition, C and $\Psi(t)$ are $2^{k-1}(M+1) \times 1$ matrices given by

$$C = [c_{10}, c_{11}, \ldots, c_{1M}, c_{20}, \ldots, c_{2M}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}M}]^T,$$

$$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \ldots, \psi_{1M}(t), \psi_{20}(t), \ldots, \psi_{2M}(t), \ldots, \psi_{2^{k-1}0}(t), \ldots, \psi_{2^{k-1}M}(t)]^T.$$

3 OPERATIONAL MATRIX OF THE FRACTIONAL INTEGRATION

In this section, we derive the Bernoulli wavelet operational matrix of the fractional integration. The integration of

$$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \ldots, \psi_{1M}(t), \psi_{20}(t), \ldots, \psi_{2M}(t), \ldots, \psi_{2^{k-1}0}(t), \ldots, \psi_{2^{k-1}M}(t)]^T$$

can be approximated by $\int_0^t \Psi(t')dt' \approx P\Psi(t)$, where P is called the Bernoulli wavelet operational matrix of integration. To derive the Bernoulli wavelet operational matrix of the general order of integration, we recall the fractional integral of order $q(>0)$ which is defined by (1).

The Bernoulli wavelet operational matrix $P^{(q)}$ for integration of the general order q is given by

$$\begin{aligned} P^{(q)}\Psi(t) &\approx I^q \Psi(t) \\ &= [I^q\psi_{10}(t), I^q\psi_{11}(t), \ldots, I^q\psi_{1M}(t), I^q\psi_{20}(t), \ldots, I^q\psi_{2M}(t), \ldots, I^q\psi_{2^{k-1}0}(t), \ldots, I^q\psi_{2^{k-1}M}(t)]^T. \end{aligned}$$

4 THE OPERATIONAL MATRIX OF MULTIPLICATION

The following property of the product of two Bernoulli wavelet function vectors will also be used:

$$A^T \Psi(t)\Psi(t)^T \approx \Psi(t)^T \tilde{V}^T, \tag{4}$$

where $\tilde{V}$ is a $2^{k-1}(M+1)\times 2^{k-1}(M+1)$ multiplication operational matrix. To illustrate the calculation procedure, consider $A^T\Psi(t)\Psi(t)^T$ given in following:

$$A^T\Psi(t)\Psi(t)^T = [\upsilon_1(t),\ldots,\upsilon_{2^{k-1}(M+1)}(t)].$$

Now we approximate $A^T\Psi(t)\Psi(t)^T$ by $\Psi(t)$ as:

$$\upsilon_i(t) \approx \tilde{\upsilon}_{i10}\psi_{10}(t)+\ldots+\tilde{\upsilon}_{i2^{k-1}M}\psi_{2^{k-1}M}(t) = \sum_{r=1}^{2^{k-1}}\sum_{s=0}^{M}\tilde{\upsilon}_{irs}\psi_{rs}(t) = \tilde{V}_i^T\Psi(t), \tag{5}$$

where

$$\tilde{V}_i = [\tilde{\upsilon}_{i10},\ldots,\tilde{\upsilon}_{i2^{k-1}M}]^T.$$

Using Eq. (5) we obtain

$$\upsilon_i^{jk} = <\sum_{r=1}^{2^{k-1}}\sum_{s=0}^{M}\tilde{\upsilon}_{irs}\psi_{rs}(t),\psi_{jk}(t)> = \sum_{r=1}^{2^{k-1}}\sum_{s=0}^{M}\tilde{\upsilon}_{irs}d_{rs}^{jk},\ \ j=1,2,\ldots,2^{k-1}, k=0,1,\ldots,M,$$

that $\upsilon_i^{jk} = <\upsilon_i(t),\psi_{jk}(t)>$, $d_{rs}^{jk} = <\psi_{rs}(t),\psi_{jk}(t)>$. Therefore

$$V_i^T = \tilde{V}_i^T D,$$

where

$$V_i = [\upsilon_i^{10},\upsilon_i^{11},\ldots,\upsilon_i^{1M},\upsilon_i^{20},\ldots,\upsilon_i^{2M},\ldots,\upsilon_i^{2^{k-1}0},\ldots,\upsilon_i^{2^{k-1}M}]^T,$$

and

$$D = [d_{rs}^{jk}],$$

where D is a matrix of order $2^{k-1}(M+1)\times 2^{k-1}(M+1)$ and is given by

$$D = \int_0^1 \Psi(t)\Psi(t)^T dt.$$

Therefore $\tilde{V}_i^T$ in Eq. (5) is given by

$$\tilde{V}_i^T = V_i^T D^{-1}.$$

Therefore we achieve the operational matrix of multiplication as:

$$\tilde{V} = [\tilde{\upsilon}_{irs}];\quad 1\le i\le 2^{k-1}(M+1),\quad 1\le r\le 2^{k-1},\quad 0\le s\le M.$$

5 PROBLEM STATEMENT

Consider the following fractional optimal control problem

$$J = \frac{1}{2}\int_0^1[u(t)^T u(t) + x(t)^T x(t)]dt,$$

where

$$x(t) = [x_1(t), x_2(t),\ldots,x_r(t)]^T,$$

$$u(t) = [u_1(t), u_2(t),\ldots,u_s(t)]^T,$$

are state and control vectors. Subject to dynamical system

$$D^q x(t) = a(t)x(t) + b(t)u(t),\quad 0<q\le 1, \tag{6}$$

with condition

$$x(0) = x_0,$$

where $a(t)$ and $b(t)$ are continuous matrix functions of time as follows

$$a(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1r}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2r}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{r1}(t) & a_{r2}(t) & \dots & a_{rr}(t) \end{bmatrix}, \quad b(t) = \begin{bmatrix} b_{11}(t) & b_{12}(t) & \dots & b_{1s}(t) \\ b_{21}(t) & b_{22}(t) & \dots & b_{2s}(t) \\ \vdots & \vdots & \vdots & \vdots \\ b_{s1}(t) & b_{s2}(t) & \dots & b_{ss}(t) \end{bmatrix},$$

and x_0 is specified constant vector. We expand the fractional state rate $D^q x_k(t)$ and control variable $u_k(t)$ with the Bernoulli wavelets as

$$D^q x_k(t) \approx X_k^T \Psi(t), \tag{7}$$

$$u_k(t) \approx U_k^T \Psi(t), \tag{8}$$

where

$$X_k = [x_{10}^k, \dots, x_{2^{k-1}M}^k]^T,$$

$$U_k = [u_{10}^k, \dots, u_{2^{k-1}M}^k]^T,$$

are unknown. Using Eq. (2), $x(t)$ can be represented as

$$x_k(t) = I^q D^q x_k(t) + x_k(0) \approx (X_k^T P^{(q)} + d_k^T)\Psi(t), \tag{9}$$

where $P^{(q)}$ is the fractional operational matrix of integration of order q and

$$x_k(0) \approx d_k^T \Psi(t).$$

Also, we expand other functions of t in the problem with respect to Bernoulli wavelets as

$$a_{ij}(t) \approx A_{ij}^T \Psi(t), \quad b_{ij}(t) \approx B_{ij}^T \Psi(t), \tag{10}$$

where

$$A_{ij} = [a_{10}^{ij}, \dots, a_{2^{k-1}M}^{ij}]^T,$$

$$B_{ij} = [b_{10}^{ij}, \dots, b_{2^{k-1}M}^{ij}]^T.$$

By replacing the approximation of x(t) and u(t) in the performance index J, this functional can be calculated numerically.

Using Eqs. (6)–(10) the dynamical system (6) can also be approximated as

$$X_k^T \Psi(t) - \sum_{i=1}^{r} a_{ki}(t) x_i(t) - \sum_{j=1}^{s} b_{kj}(t) u_j(t) = 0,$$

or

$$X_k^T \Psi(t) - \sum_{i=1}^{r} A_{ki}^T \Psi(t)\Psi(t)^T (X_i^T P^{(q)} + d_i^T)^T - \sum_{j=1}^{s} B_{kj}^T \Psi(t)\Psi(t)^T U_j = 0.$$

Now using Eq. (4) we have:

$$A_{kr}^T \Psi(t)\Psi(t)^T \approx \Psi(t)^T \tilde{A}_{kr}^T, \quad B_{ks}^T \Psi(t)\Psi(t)^T \approx \Psi(t)^T \tilde{B}_{ks}^T,$$

then, we obtain

$$X_k^T \Psi(t) - \sum_{i=1}^{r} \Psi(t)^T \tilde{A}_{ki}^T (X_i^T P^{(q)} + d_i^T)^T - \sum_{j=1}^{s} \Psi(t)^T \tilde{B}_{kj}^T U_j = 0,$$

or

$$H_k = X_k^T - \sum_{i=1}^{r} (X_i^T P^{(q)} + d_i^T) \tilde{A}_{ki} - \sum_{j=1}^{s} U_j^T \tilde{B}_{kj}, \quad k = 1, \ldots, r.$$

Let

$$J^* = J + \sum_{k=1}^{r} H_k \lambda_k,$$

where

$$\lambda_k = [\lambda_{10}^k, \ldots, \lambda_{2^{k-1}M}^k]^T,$$

are the unknown Lagrange multipliers. Now the necessary conditions for the extremum are

$$\frac{\partial J^*}{\partial X_i} = 0, \quad i = 1, \ldots, r$$

$$\frac{\partial J^*}{\partial U_j} = 0, \quad j = 1, \ldots, s$$

$$\frac{\partial J^*}{\partial \lambda_k} = 0 \quad k = 1, \ldots, r.$$

As a result, by replacing the values obtained from solving of above system using Newton's iterative method in Eq. (8) and (9), u(t) and x(t) can be calculated.

6 ILLUSTRATIVE EXAMPLE

To demonstrate the effectiveness of the method, in this section we present numerical results. We consider the following problem: Find the control $u(t)$ which minimizes the quadratic performance index (Rabiei et al., 2017)

$$J = \frac{1}{2} \int_0^1 [x_1^2(t) + x_2^2(t) + u^2(t)] dt,$$

subject to the system dynamics

$$D^q x_1(t) = -x_1(t) + x_2(t) + u(t),$$

$$D^q x_2(t) = -2x_2(t),$$

and the initial condition

$$x_1(0) = 1, \; x_2(0) = 1.$$

The closed form solution for this system for $q = 1$ is given as

$$x_1(t) = \frac{-3}{2} e^{-2t} + 2.48164 e^{-\sqrt{2}t} + 0.018352 e^{\sqrt{2}t},$$

$$x_2(t) = e^{-2t},$$

$$u(t) = \frac{1}{2} e^{-2t} - 1.02793 e^{-\sqrt{2}t} + 0.0443056 e^{\sqrt{2}t},$$

and

$$J = 0.43198.$$

Tables 1, 2 and 3 show the comparison between the absolute error of $x_1(t)$, $x_2(t)$ and $u(t)$ for when $q=1$, and different values of M and k. It is obvious from these tables, that with increase in the number of the Bernoulli wavelet basis, the approximate values of $x_1(t)$, $x_2(t)$ and $u(t)$ converge to the exact solutions. Figures 1, 2 and 3 demonstrate the approximation of $x_1(t)$, $x_2(t)$ and $u(t)$ for different values of q together with the exact solution for $q=1$. It can be observed that for $q=1$ the numerical solution agrees with the analytical solution. Thus, as q approaches to 1, the solution for the integer order system is recovered. Also, in Table 4, the results of J for different values of q with $k=2, M=3$ are listed.

Table 1. Absolute error of $x_1(t)$ for $q=1$.

	M=4		M=6	
t	$k=1$	$k=2$	$k=1$	$k=2$
0.1	0.0007374747	0.0001042201	8.06599×10^{-6}	7.13609×10^{-6}
0.2	0.0009037144	0.000082740 6	0.0000143549	5.90006×10^{-6}
0.3	8.96865×10^{-6}	0.0000582562	0.0000156977	5.14809×10^{-6}
0.4	0.0007589444	0.0001147822	6.45609×10^{-7}	4.66307×10^{-6}
0.5	0.0008752991	0.0000697113	8.25493×10^{-6}	4.23078×10^{-6}
0.6	0.0003222922	0.0000217926	2.27843×10^{-6}	3.42673×10^{-6}
0.7	0.00053504	0.000021465	0.0000152999	2.8608×10^{-6}
0.8	0.00104752	0.0000135519	7.56901×10^{-6}	2.4722×10^{-6}
0.9	0.00036008	0.0000255926	0.0000139419	2.18291×10^{-6}

Table 2. Absolute error of $x_2(t)$ for $q=1$.

	M=4		M=6	
t	$k=1$	$k=2$	$k=1$	$k=2$
0.1	0.00104368	0.00014466	0.0000139531	1.74618×10^{-7}
0.2	0.00140928	0.0000958637	7.57297×10^{-6}	9.49348×10^{-8}
0.3	0.0001357322	0.0000722021	9.84248×10^{-6}	5.25724×10^{-8}
0.4	0.00105829	0.0001528444	4.56867×10^{-6}	1.410017×10^{-7}
0.5	0.00133809	0.0001351855	0.0000112909	3.39221×10^{-7}
0.6	0.0005927078	0.0000532298	1.22927×10^{-6}	6.42383×10^{-8}
0.7	0.0006962669	0.0000352563	0.0000114135	3.49246×10^{-8}
0.8	0.00156213	0.0000265535	4.89582×10^{-6}	1.93403×10^{-8}
0.9	0.000640961	0.0000562349	0.0000148349	5.18712×10^{-8}

Table 3. Absolute error of $u(t)$ for $q=1$.

	M=4		M=6	
t	$k=1$	$k=2$	$k=1$	$k=2$
0.1	0.000228783	0.0000310729	3.02849×10^{-6}	2.07308×10^{-6}
0.2	0.000311844	0.0000242102	4.47929×10^{-6}	1.71941×10^{-6}
0.3	0.0000304535	0.0000175621	5.08176×10^{-6}	1.51631×10^{-6}
0.4	0.00023299	0.00003394	2.93708×10^{-7}	1.39739×10^{-6}
0.5	0.000295343	0.0000345583	2.87759×10^{-6}	1.31117×10^{-6}
0.6	0.000132324	0.0000122394	6.02416×10^{-7}	1.07019×10^{-6}
0.7	0.000151431	9.015075×10^{-6}	5.03803×10^{-6}	9.17113×10^{-7}
0.8	0.00034376	7.89573×10^{-6}	2.59491×10^{-6}	8.23534×10^{-7}
0.9	0.000139359	0.0000128748	4.60326×10^{-6}	7.69222×10^{-7}

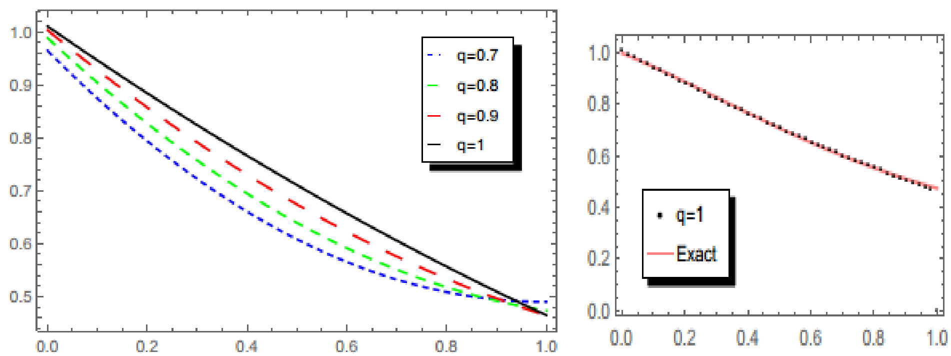

Figure 1: Comparison of $x_1(t)$ for $k=1, M=3$ and with $q=0.7,0.8,0.9,1$ and exact solution

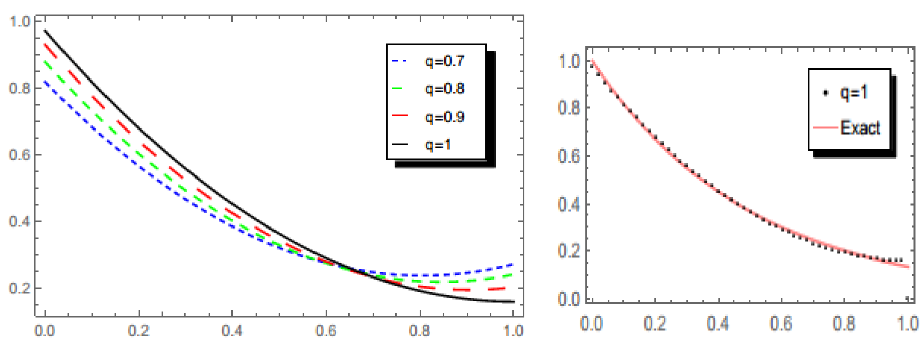

Figure 2: Comparison of $x_2(t)$ for $k=1, M=3$ and with $q=0.7,0.8,0.9,1$ and exact solution

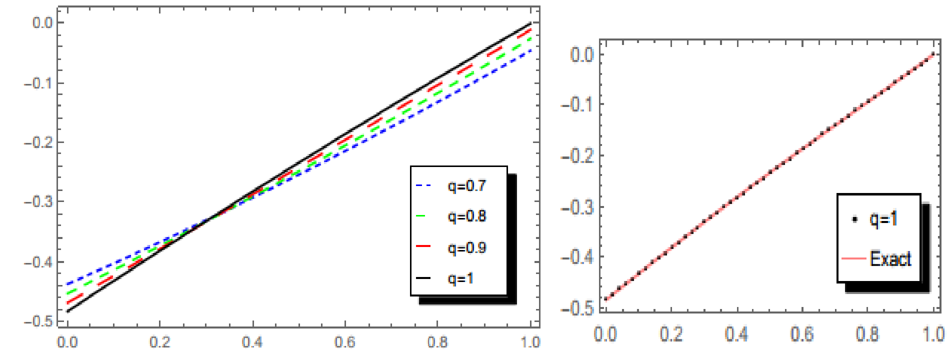

Figure 3: Comparison of $u(t)$ for $k=1, M=3$ and with $q=0.7,0.8,0.9,1$ and exact solution

Table 4. The results of J for different values of q.

	$q=0.7$	$q=0.8$	$q=0.9$	$q=1$
J	0.351576	0.376254	0.40308	0.431987

7 CONCLUSION

In this paper, a general formulation for the Bernoulli wavelet operational matrix of fractional order integration and multiplication has been derived. These matrices is used to approximate numerical solution of fractional optimal control problems. The achieved results are compared with exact solutions. Results showed that as the value of q approaches 1, the numerical solutions for both the state and the control variables approach to the analytical solutions for $q=1$.

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