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The Maximum Edge Eccentricity Energy of a Graph

Akram Sadat Banihashemi Dehkordi, Saeed Mohammadian Semnani
Semnan University
Semnan University, Semnan, Iran
Banihashemi.akram@semnan.ac.ir, S_mohammadian@semnan.ac.ir

ABSTRACT

In this paper, we introduce the concept of a maximum edge eccentricity matrix M_{e_e}(G) of a connected graph G and obtain some coefficients of the characteristic polynomial P(G,υ) of the maximum edge eccentricity matrix of G. We also introduce the maximum edge eccentricity energy EM_{e_e}(G) of a connected graph.

KEYWORDS: Edge distance in graph, maximum edge eccentricity matrix, maximum edge eccentricity eigenvalue, maximum edge eccentricity energy of a graph

1 INTRODUCTION

In this paper, all graphs are assumed to be simple, finite and connected. A graph G= (V, E) is a simple graph, that is, having no loops, no multiple and directed edges. As usual, we denote by n=$|V|$ and m=$|E|$ to the number of vertices and edges in a graph G , respectively. For an Edge e∈ E, the open neighborhood of e in a graph G, denoted N(e), is the set of all edges that are adjacent to e. The edge distance d(e, e') between any two edges e and e'in a graph G is the minimum number of edges between e and e'. For an edge e of G, the edge eccentricity is e(e) = max {d(e,e') ; $e' \in E(G)$}. The concept energy of a graph introduced by I. Gutman [4], in the year 1978. Let G be a graph with n vertices and m edges and let A(G)= (a_{ij}) be the adjacency matrix of G, where

$$a_{ij} = \begin{cases} 1 & if \;\; v_i v_j \in E(G) \\ 0 & otherwise \end{cases}$$

The eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of a matrix A(G), assumed in non-increasing order, are the eigenvalues of the graph G. The energy E(G) of a graph G is defined to be the sum of the absolute values of the eigenvalues of G, i.e.,$E(G) = \sum_{i=1}^{n} |\lambda_i|$. Prof. C. Adiga and M. Smitha [5], have defined the maximum degree energy EM(G) of a graph G which depend on the maximum degree matrix M(G) of G. Let G be a simple graph with n vertices $\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_\mathrm{n}$. Then the maximum degree matrix M(G) of a graph G defines as, where

$$d_{ij} = \begin{cases} \max\{d(v_i), d(v_j)\} & \text{if } v_iv_j \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

As M(G) is real symmetric with zero trace, then the eigenvalues of G being real with sum equal to zero. Motivated by this paper, we introduce the concept of a maximum edge eccentricity matrix $M_{e_e}(G)$ of a connected graph G and obtain some coefficients of the characteristic polynomial P(G,υ) of the maximum edge eccentricity matrix of G. We also introduce the maximum eccentricity energy $EM_{e_e}(G)$ of a connected graph G.

2 THE MAXIMUM EDGE ECCENTRICITY ENERGY OF A GRAPH

Definition: For an edge e_i of G, the edge eccentricity is $e(e_i) = \max\{d(e_i, e_j);\ e_j \in E(G)\}$ that the edge distance $d(e_i, e_j)$ in a graph G is the minimum number of edges between e_i and e_j. Let G(V, E) be a simple connected graph with m edge $e_1, e_2, \cdots, e_m$ and let $e(e_i)$ be the eccentricity of edge e_i, $i = 1,2, \cdots, m$. The Maximum edge eccentricity matrix of G defining as,

$$M_{e_e}(G) = \begin{pmatrix} {e^*}_{11} & {e^*}_{12} & \cdots & {e^*}_{1m} \\ {e^*}_{21} & {e^*}_{22} & \cdots & {e^*}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ {e^*}_{m1} & {e^*}_{m2} & \cdots & {e^*}_{mm} \end{pmatrix}$$

where

$${e^*}_{ij} = \begin{cases} \max\{ e(e_i), e(e_j)\} & if\ \ e_i, e_j\ are\ adjacent \\ 0 & if\ \ i = j \\ 0 & otherwise \end{cases}$$

The characteristic polynomial of the maximum edge eccentricity matrix M_{e_e}(G) is defined by P(G,υ)= $\det(\upsilon I - M_{e_e}(G))$. Where I is the unit matrix of order m. The maximum edge eccentricity eigenvalues of G are the eigenvalues of M_{e_e}(G). Since M_{e_e}(G) is real and symmetric with zero trace, then its eigenvalues are real numbers with sum equals to zero. We label them in non-increasing order $\upsilon_1 \geq \upsilon_2 \geq \cdots \geq \upsilon_m$. The maximum edge eccentricity energy of a graph G is defined as

$$EM_{e_e}(G) = \sum_{i=1}^{m} |\upsilon_i|$$

To illustrate this concept, we study the following examples.

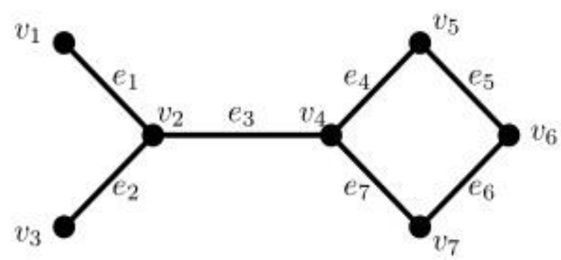

Figure 1: Graph G_1

2.1 Example 1

let G_1be a graph in figure 1, with 7 edge $e_1, e_2, e_3, e_4, e_5, e_6$ and e_7. The edge distance matrix of G_1 is

$$M_{e_e}(G_1) = \begin{pmatrix} 0 & 2 & 2 & 0 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & 2 & 0 \end{pmatrix}$$

The characteristic polynomial of $M_{e_e}(G_1)$ is P(G_1,υ)= det$\left(\upsilon I - M_{e_e}(G_1) \right)$

$$= \begin{vmatrix} \upsilon & -2 & -2 & 0 & 0 & 0 & 0 \\ -2 & \upsilon & -2 & 0 & 0 & 0 & 0 \\ -2 & -2 & \upsilon & -1 & 0 & 0 & -1 \\ 0 & 0 & -1 & \upsilon & -2 & 0 & -1 \\ 0 & 0 & 0 & -2 & \upsilon & -2 & 0 \\ 0 & 0 & 0 & 0 & -2 & \upsilon & -2 \\ 0 & 0 & -1 & -1 & 0 & -2 & \upsilon \end{vmatrix}$$

$$= \upsilon^7 - 27\upsilon^5 - 18\upsilon^4 + 184\upsilon^3 + 208\upsilon^2 - 112\upsilon - 32$$

Then the maximum edge eccentricity eigenvalues of G_1 are υ_1=-3.5616, υ_2= -2.5251, υ_3=-2, υ_4=-0.2159, υ_5=0.5616, υ_6=3.3159, υ_7=4.4251. The maximum edge eccentricity energy of G_1 is

$$EM_{e_e}(G_2) = 16.6052$$

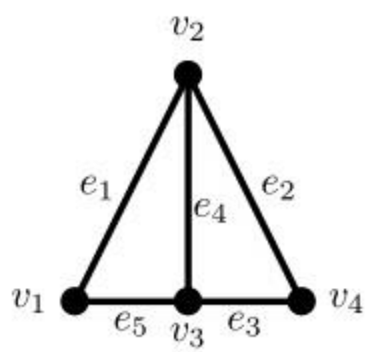

Figure 2: Graph G_2

2.2 Example 2

Let G_2 be the graph in figure 2, with 5 edge e_1, e_2, e_3, e_4 and e_5. The maximum edge eccentricity matrix of G_2 is

$$M_{e_e}(G_2) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic polynomial of $M_{e_e}(G_2)$ is $P(G_2,\upsilon)= \det\left(\upsilon I - M_{e_e}(G_2) \right)$

$$= \begin{vmatrix} \upsilon & -1 & -1 & 0 & -1 \\ -1 & \upsilon & -1 & -1 & -1 \\ -1 & -1 & \upsilon & -1 & 0 \\ 0 & -1 & -1 & \upsilon & -1 \\ -1 & -1 & 0 & -1 & \upsilon \end{vmatrix}$$

$$= \upsilon^5 - 8\,\upsilon^3 - 8\upsilon^2$$

Then the maximum edge eccentricity eigenvalues of G_2 are υ_1=-2, υ_2=-1.2361 , υ_3=0, υ_4=0, υ_5=3.2361. The maximum edge eccentricity energy of G_2 is

$$EM_{e_e}(G_2) = 6.4722$$

3 PROPERTIES OF MAXIMUM EDGE ECCENTRICITY ENERGY

In this section, we obtain the values of some coefficients of the characteristic polynomial of the maximum edge eccentricity matrix and investigate some properties of maximum edge eccentricity of a graph G.

3.1 Theorem 1

Let G be a graph of order m and let $P(G.\upsilon) = c_0\upsilon^m + c_1\upsilon^{m-1} + \cdots + c_m$ be the characteristic polynomial of maximum edge eccentricity matrix of G. Then

i. $\mathbf{c_0 = 1}$

ii. $c_1 = 0$

iii. $c_2 = -\sum_{1\le i<j<k\le m}(e^*{}_{ij})^2$

iv. $c_3 = -2\sum_{1\le i<j<k\le m} e^*{}_{ij}e^*{}_{ik}e^*{}_{jk}$

v. $c_m = (-1)^m \det(M_{e_e})$

3.1.1 Proof

i. Directly from the definition of $P(G,\upsilon)$, it follows that $c_0 = 1$.

ii. Since the sum of diagonal elements of $M_{e_e}(G)$ is equal to 0 sum of determinants of all 1*1 principal submatrices of $M_{e_e}(G)$ is the trace of $M_{e_e}(G)$ which evidently equal to 0. Thus $c_1 = 0$.

iii. $(-1)^2c_2$ is equal to sum of determinats of all the 2*2 principal submatrices of $M_{e_e}(G)$, that is

$$c_2 = \sum_{1\le i<j\le m}\begin{vmatrix} e^*{}_{ii} & e^*{}_{ij} \\ e^*{}_{ji} & e^*{}_{jj}\end{vmatrix} = \sum_{1\le i<j\le m}(e^*{}_{ii}e^*{}_{jj} - e^*{}_{ij}e^*{}_{ji})$$

$$= \sum_{1\le i<j\le m} e^*{}_{ii}e^*{}_{jj} - \sum_{1\le i<j\le m}\left(e^*{}_{ij}\right)^2 = 0 - \sum_{1\le i<j\le m}\left(e^*{}_{ij}\right)^2$$

$$= -\sum_{1\le i<j\le m}\left(e^*{}_{ij}\right)^2$$

iv. $$c_3 = (-1)^3 \sum_{1\le i<j<k\le m}\begin{vmatrix} e^*{}_{ii} & e^*{}_{ij} & e^*{}_{ik} \\ e^*{}_{ji} & e^*{}_{jj} & e^*{}_{jk} \\ e^*{}_{ki} & e^*{}_{kj} & e^*{}_{kk}\end{vmatrix}$$

$$= \sum_{1\le i<j<k\le m}[e^*{}_{ii}\left(e^*{}_{jj}\,e^*{}_{kk} - e^*{}_{kj}\,e^*{}_{jk}\right) - e^*{}_{ij}\left(e^*{}_{ji}e^*{}_{kk} - e^*{}_{ki}\,e^*{}_{jk}\right)$$
$$+ e^*{}_{ik}\left(e^*{}_{ji}e^*{}_{kj} - e^*{}_{ki}\,e^*{}_{jj}\right)]$$

$$= -\sum_{1\le i<j<k\le m} e^*_{ii}\, e^*_{jj}\ e^*_{kk} + \sum_{1\le i<j<k\le m} [\, e^*_{ii}(e^*_{jk})^2 + e^*_{jj}\,(e^*_{ik})^2 + \ e^*_{kk}\,(e^*_{ij})^2]$$
$$- \sum_{1\le i<j<k\le m} e^*_{ij}e^*_{jk}e^*_{ki} - \sum_{1\le i<j<k\le m} e^*_{ik}e^*_{kj}e^*_{ji}$$
$$= -0 + 0 - \ 2 \sum_{1\le i<j<k\le m} e^*_{ij}e^*_{ik}e^*_{jk}$$

Thus

$$c_3 = \ -2 \sum_{1\le i<j<k\le m} e^*_{ij}e^*_{ik}e^*_{jk}$$

v. The proof is obvious.

Note : In Theorem 1, we can write c_2 as follows:

$c_2 = \ - \sum_{i=1}^{m}(x_i + y_i)e^2(\,e_i)$, where

$x_i = \ |\{\, e' \in N(e_i) \colon e(e') < e(e_i), 1 \le i \le m\,\}|$ and

$y_i = \ \left|\{\, e_j \in N(e_i),\ j > i : e(e_j) = e(e_i),\ 1 \le i \le m\,\}\right|$.

3.2 **Theorem 2**

Let $v_1, v_2, \cdots, v_m$ be the maximum edge eccentricity eigenvalues of $M_{e_e}(G)$. Then

i. $\sum_{i=1}^{m} v_i = 0$

ii. $\sum_{i=1}^{m} {v_i}^2 = 2\ \sum_{1\le i<j<k\le m}(e^*_{ij})^2$

iii. $\sum_{i=1}^{m} {v_i}^3 = 6\ \sum_{1\le i<j<k\le m} e^*_{ij}e^*_{ik}e^*_{jk}$

3.2.1 Proof

The proof is consequences of Newton's identity and Theorem 1.

3.3 Theorem 3

If the maximum edge eccentricity energy of a graph G is rational, then it must be an even integer.

3.3.1 Proof

Let G be a graph of order m and $v_1, v_2, \cdots, v_m$ be the maximum edge eccentricity eigenvalues of G. since, $\sum_{i=1}^{m} v_i = 0$, then $v_1, v_2, \cdots, v_r$ be the positive eigenvalues of G and the remaining are non positive. Then

$$\mathrm{EM}_{\mathrm{e_e}} = \upsilon_1 + \upsilon_2 + \cdots + \upsilon_r - (\upsilon_{r+1} + \upsilon_{r+2} + \cdots + \upsilon_m)$$

$$= 2(\upsilon_1 + \upsilon_2 + \cdots + \upsilon_r)$$

Since, $\upsilon_1, \upsilon_2, \cdots, \upsilon_r$ are algebraic numbers, so is their sum and hence must be integer if EM_{e_e} is rational.

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Email: banihashemi.akram@semnan.ac.ir
Email: s_mohammadian@semnan.ac.ir