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FUZZY K-IDEAL IN BCI-ALGEBRAS

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ABSTRACT

In this paper with the notion of fuzzy k-ideal of BCI-algebras, we discuss and review several results described in BCI-algebras.

KEYWORDS: BCI-algebra, k-deal, fuzzy ideal, fuzzy a-ideal.

1 INTRODUCTION

The class of BCI-algebras which introduced by Iseki [7] is an important class of logical algebras which has two origins. One of the motivations for their use in based on set theory, the other o classical and non-classical propositional calculus. The concept of fuzzy sets introduced by Zadeh [8] was applied to BCI-algebra by Xi[3].Since then many researchers have investigated various properties of these algebras.Chun.et al.[2] constructed an extension of a k- semi ring and studied a k-ideal of a k-semi ring.

2 PRELIMINARIES

First we present the fundamental definition. By a BCI-algebra we mean a nonempty set X with a binary operation $*$ and constant 0 satisfying the following conditions

(i) $((x * y) * (x * z)) * (z * y) = 0$,
(ii) $(x * (x * y)) * y = 0$,
(iii) $x * x = 0$,
(iv) $x * 0 = 0$,
(v) $x * y = 0$ and $y * x = 0$ imply that $x = y$ for all $x, y, z \in X$.

A partial ordering "$\leq$" on X can be defined by $x \leq y$ if and only if $x * y = 0$.A nonempty subset S of a BCI-algebra X is called a subalgebra of X if $x * y \in S$ whenever $x, y \in S$.

Definition 2.1: A nonempty subset I of a BCI-algebra X is called an ideal of X if;

(i) $0 \in I$, for all $x \in X$
(ii) $(x * y) \in I$ and $y \in I$ imply that $x \in I$ for all $x, y, \in X$.

By a fuzzy set μ in a nonempty set X we mean a function μ: X → [0, 1].
A mapping f: X → Y of BCI-algebras is called homomorphism if f(x ∗ y) = f(x) ∗ f(y), for all x ∈ X. Let Im (μ) denote the image set of μ. We will write x∧y by min {x, y}, and x∨y for max{x, y}, where x, y are real numbers. Given a fuzzy set μ and t ∈ [0, 1].
Let U (μ, t) = {x ∈ X |μ(x) ≥ t}, U (μ, t) is called level subset of μ.

Definition 2.2: Let X be a BCI-algebra. A non-empty subset I of X is called k-ideal in X if :
(i) 0 ∈ I,
(ii) (x∗ y) ∨ (y ∗ x) ∈ I and y ∈ I imply x ∈ I.

Definition 2.3: For any x, y in BCI-algebra X,
(i) if μ(x ∗ y) ≥ μ(x) ∧ μ(y) , then μ is said to be a fuzzy sub algebras.
(ii) if μ(0) ≥ μ(x) and μ(x) ≥ μ(x ∗ y) ∧ μ(y) ,then μ is called a fuzzy ideal of X.

Definition 2.4: A fuzzy set μ is called fuzzy k-ideal if for all x, y ∈ X ;
(F1) μ(0) ≥ μ(x)
(F2) μ(x) ≥ μ((x ∗ y) ∨ μ(y ∗ x)) ∧ μ(y)

3. Main Results

In what follows, let X denote a BCI-algebra unless otherwise specified.

Theorem1. A fuzzy set μ of a BCI-algebra X is a fuzzy k-ideal of X if and only if for any t ∈ [0, 1] , U(μ, t) is either empty or a k-ideal of X.
Proof. Let μ is a fuzzy k-ideal of X. Then by definition μ (0) ≥ μ(x) , for all x ∈ X .
Therefore, μ (0) ≥ μ(x) ≥ t, for all t ∈ [0, 1] or μ(0) ≥ t implies 0 ∈ U(μ, t) .
Let (x ∗ y) or (y ∗ x) ∈ U (μ, t) .Then μ(x ∗ y) ≥ t or μ(y) ≥ t. Then (μ(x ∗ y) ∨ (y ∗ x)) ≥ t and μ(y) ≥ t. Since μ is a fuzzy k-ideal of X, therefore for all x, y ∈ X,
μ (x) ≥ (μ(x ∗ y) ∨ μ(y ∗ x)) ∧ μ(y) ≥ t or μ(x) ≥ t implies x ∈ U(μ, t). This proves that the level Subset a fuzzy k-ideal in X. is k-ideal of X.
Conversely, we show that μ is a fuzzy k-ideal of X. Assume U (μ, t) is a k-ideal of X, for any t ∈ [0, 1] , with U(μ, t) ≠ ∅. For any x, y ∈ X, let μ(x) =t_1 , μ(x ∗ y) = t_2, μ(x) = t_3 , (t_i ∈ [0, 1]. If we let t = min {max{t_1, t_3} t_1} . Then y ∈ U (μ, t) and y ∗ x ∈ U(μ, t) or x ∗ y ∈ U(μ, t) . Since U(μ, t) is a K-ideal of X, we have x ∈U (μ, t).
i.e, μ(x) ≥ (μ(x ∗ y) ∨ μ(y ∗ x)) ∧ μ(y) . Hence μ is a fuzzy k-ideal in X.

Theorem2. Let μ be a fuzzy set in BCI-algebra X. If μ is a fuzzy k-ideal of X, then the set J = {x ∈ X| μ(x) = μ(0)} , is a k-ideal of X.
Proof. Assume that μ is a fuzzy k-ideal of X, clearly 0 ∈ J. Let (x∗ y) or (y∗ x) ∈ J and Y ∈ J.
Since μ is a fuzzy k-ideal of X, for all x, y ∈ X,

M (x) ≥ (μ(x ∗ y) ∨ μ(y ∗ x)) ∧ μ(y) = (μ(0) ∨ μ(0)) ∧ μ(0) = μ(0) .

Or μ (0) ≥ μ(x) , but μ(x) < μ(0) . Hence μ (0) = μ(x) or x ∈ J . This proves that J is a k-ideal in X.

Theorem3. Let I be a k-ideal of X. Then there exists a fuzzy k-ideal μ of X such that U(μ, t) = I .
Proof. If we define a fuzzy subset of X by μ(x) = $\begin{cases} t, & if \ x \epsilon X \\ 0, & otherwise \end{cases}$
For some t ∈ [0,]. Then it follows that U (μ, t) = I . For a given s ∈ [0, 1], we have

$$\mu(x) = \begin{cases} U(\mu, 0)(= X)\ if\ s = 0, \\ U(\mu, t) \quad (= I)\ if s \le t, \\ \emptyset \qquad if\ t < s \le 1 \end{cases}$$

Since I and X itself are k-ideals of X, it follows that every non-empty subset U (μ, s) of μ is k-ideal of X. By Theorem3.1 μ is a fuzzy k-ideal of X, proving the theorem.

Theorem4. Let μ be a fuzzy k-ideal of X .Then two level subset k-ideal U (μ, s), U (μ, t) (with s < t in [0, 1]) of μ are equal if and only if there is no x ∈ X such that s ≤ μ(x) ≤ t .
Proof. Suppose s < t in [0, 1] and U (μ, s) = U (μ, t) . If there exists x ∈ X, such that s ≤ μ(x) ≤ t, then U (μ, t) is a proper subset of U(μ , s) , a contradiction .
Conversely, suppose that there is no x ∈ X such that s ≤ μ(x) ≤ t. Note that s < t implies U (μ, t) ⊆ U(μ, s) .If x ∈ U(μ, s), then μ(x) ≥ s , and μ(x) ≥ t , because μ < t . Hence x ∈ U (μ, t) and U (μ, s) = U(μ, t) .This completes the proof. Given a fuzzy k-ideal of X, we denote by by Im (μ) the image set of μ.

Theorem5. Let μ be a fuzzy k-ideal of X. If Im (μ) = $\{t_1, t_2, \cdots, t_n\}$where $t_1 < t_2 < \cdots t_n$,then the family of k-ideals U(μ, t_i) (i = 1, 2,…, n) constitutes the collection of all level ideals of μ.
Proof. Let f: X → Y be an onto homomorphism. Let ν be a fuzzy k-ideal on Y If t ∈ [0, 1] with t < t_1 , then U(μ, t_1) ⊆ U(μ, t) .Since U(μ, t_1) = X , we have U(μ, t) = X and U(μ, t) = U(μ, t_1) . If t ∈ [0, 1] with t_i < t < t_{i+1}(1 ≤ i ≤ n − 1), then there is no x ∈ X such that t < μ(x) < t_{i+1}. It follows from Theorem 3.4 that U(μ, t) = U(μ, t_{i+1}) . This shows that for any t ∈ [0, 1] with t < μ (0) , the level ideal U(μ, t) is in {U(μ, t_i}), 1 ≤ i ≤ n. This completes the proof.

Given any two sets X and Y, let μ be a fuzzy subset of BCI-algebra X, and let f: X → Y be any mapping .We define a fuzzy subset ν on Y by

$$\vartheta(y) = \begin{cases} sup\ \mu(x) & if f^{-1}(y), y \in Y, x \in f^{-1}(y) \\ 0 & otherwise \end{cases}$$

and we call ϑ the image of μ under f , written f(μ). For any fuzzy subset ϑ on f(x), we define a Subset μ on X, by μ(x) = ϑ (f(x)) for all x ∈ X, and we call μ the pre image of ϑ under which is denoted by f^{-1}(ϑ).

Theorem6. An onto holomorphic pre image of fuzzy k-ideal is a fuzzy k-ideal.

Proof. Let f: X → Y be an onto homomorphism. Let ν be a fuzzy k-ideal on Y and let μ be Pre image of ν under f. Then it was proved that μ is a fuzzy ideal of X. For any x, y ∈ X, we have,

μ(x) = ν(f(x)) ≥ (ν(f(x) ∗ f(y) ∨ ν(f(y) ∗ f(x))) ∧ ν(f(y))= (ν(f(x ∗ y) ∨ ν(f(y ∗ x))) ∧ ν(f(y))
= (μ(x ∗ y) ∨ μ(y ∗ x)) ∧ μ(y).
Proving that μ is a fuzzy k-ideal of X.

Theorem7, [4]. Let f be a mapping from a set X to a set Y and let μ be a fuzzy subset of X. Then for every t ∈ [0, 1]

$$(f(\mu))_t = \bigcap_{0<s<t} f(_{\mu_{t-s}})$$

Theorem8. Let f: X → Y be an onto homomorphism. Let μ be a fuzzy k-ideal on X. Then the holomorphic image f(μ) is fuzzy k-ideal of Y .

Proof. In view of theorem 3.1, it is sufficient be show that each non-empty level subset of f(μ) is k- Ideal of Y. Let $(f(\mu))_t$ be a non-empty level subset of f (μ) for every t ∈ [0, 1] .
If t = 0 then $(f(\mu))_t$= Y. Assume t ≠ 0, By Theorem 3.7, $(f(\mu))_t = \bigcap_{0<s<t} f(\mu_{t-s})$
Hence $f(\mu)_t$is non-empty for each 0 < s < t , and so $\mu_{t-s}$is a non-empty level of μ for every 0 < s < t .Since μ is a fuzzy k-ideal of X. Since f is an onto homomorphism. $f(\mu_{t-s})$ is a k-ideal of Y .Hence $(f(\mu))_t$ being an intersection of a family of k-ideals is also a k-ideal of Y.
The proof is complete.

Definition 3.9: A k-ideal I of BCI-algebra X is said to be characteristic if f (I) = I for all f ∈ Aut(X) where Aut(X) is the set of all autoorphism of X . A fuzzy k-ideal μ of X is said to be a fuzzy characteristic if μ(f(x)) = μ(x) for all x ∈ X and f ∈ Aut(X).

Theorem 3.10. Let μ be a fuzzy k-ideal of BCI-algebras X, and let f: X → Y be an onto homomorphism. Then the mapping μ^f : X → [0, 1], defined by μ^f(x) = μ(f(x)) for all x ∈ X, is a fuzzy k-ideal 0f X.

Proof. It was proved that μ^f is a fuzzy k-ideal of X, for any x, y ∈ X, we have;

μ^f (x) = μ (f(x)) ≥ (μ(f(x) ∗ f(y) ∨ μ(f(y) ∗ f(x))) ∧ μ(f(y))
= (μ (f(x ∗ y) ∨ μ(f(y ∗ x))) ∧ μ(f(y))
= (μ^f (x ∗ y) ∨ μ^f (y ∗ x)) ∧μ^f (y).

Proving that μ^f is a fuzzy k-ideal of X.

Theorem 9. Let μ be a fuzzy characteristic k-ideal of BCI-algebra X, then each level k-ideal of μ is characteristic.
Proof. Let μ be a fuzzy characteristic k-ideal of BCI-algebra X. Let f ∈Aut(X). For any t ∈ [0, 1], If y ∈ $f(\mu_t)$, then μ(y) = μ(f(x)) = μ(x) ≥ t , for some x ∈ μ_t with y = f(x). It follows that y ∈ μ_t. Conversely, if y ∈ μ_t , then t < μ(y) = μ(f(x)) = μ(x) for some x ∈ X withy = f(x) . It follows that y∈f(μ_t). Hence f(μ_t) = μ_t.

Lemma1. Let μ be a fuzzy k-ideal of BCI-algebra X and Let x ∈ X, then μ(x) = t if and only if x ∈ μ_t and x∉ μ_s for all s > t.

Proof. Straight forward.

Theorem 10. Let μ be a fuzzy k-ideal of BCI-algebra X. If each level k-ideal of μ is characteristic, then μ is fuzzy characteristic.
Proof. Let x ∈ X and f ∈ Aut(X).IF μ(x) = t ∈ [0, 1], Then by lemma3.12 x ∈μ_tt and x ∉ μ_s μ_s for all s > t. Since each level k-ideal of μ is characteristic , f(x) ∈ f (μ_t) = μ_t.

Assume μ (f(x)) = s > t . Then f(x) ∈ $\mu_s = f(\mu_s)$. Since f is one-to-one, it follows that x ∈ μ_s , a contradiction. Hence μ(f(x)) = t = μ(x). Showing that μ is fuzzy characteristic .

3 CONCLUSION

In this paper with the notion of fuzzy k-ideal of BCI-algebras, we discuss and review several results described in BCI-algebras. So we can get same results in BCK-algebras.

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REFERENCES

[1] J.Meng , Xiu-e Guo : On fuzzy ideal in BCK/BCI-algebras, Fuzzy sets and system (Inpress)
[2] Chun et .al : Constructed an extension of a k-semi ring and studied a k-ideal of a k-semiring, Fuzzy Sets and Systems, 61 (1994), No. 2, 137-142.
[3] O.G.Xi : Fuzzy BCK-algebras , Math. Japon.36(1991), 935-942.
[4] Y. B. Jun, X.L.Xin : Fuzzy prime ideal and invertible fuzzy ideals in BCK-algebras, Fuzzy sets and systems. 117 (2001), 471-476.
[5] W. A. Dubek and Y. B. Jun : Quasi p-ideals of quasi BCI-algebra, Quasi groups and Related Systems, 11 (2004), 25-38.
[6] K.Hur,H.W.Kang and J.H.Ryou : Intuitionistic fuzzy generalized bi-ideals of a seimgroup, International Math. Forum, 2, (2006), No. 6, 257-274.
[7] K.Iseki : On BCI-algebras, Math. Sem. Notes 8 (1980),125-130.
[8] L. A. Zadeh : Fuzzy Sets, Information and Control, 8 (1965), 338-353..