3[TH] INTERNATIONAL CONFERENCE ON COMBINATORICS, CRYPTOGRAPHY AND COMPUTATION

Iran University of Science &Technology

Proceedings of the 3[nd] International Conference on Combinatorics, Cryptography and Computation (I4C2018)

Scalar Product Graphs of some Modules

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ABSTRACT

Let R be a commutative ring with identity and M be an R-module. The Scalar-Product Graph of M is defined as the graph $G_R(M)$ with the vertex set M and two distinct vertices x and y are adjacent if and only if there exist r belong to R that $x = ry$ or $y = rx$. In this paper, we study some relation between algebraic properties and graph concepts of $G_R(M)$.

KEYWORDS: Scalar Product, Graph, Module.

1 INTRODUCTION

The concept of the zero-divisor graph of a commutative ring, denoted by $\Gamma(R)$, was introduced by Beck [5], where he was mainly interested in coloring. $\Gamma(R)$ is graph with vertices nonzero zero divisors of R and edges those pairs of distinct nonzero zero divisors $\{a;\ b\}$ such that $ab = 0$. Let G be an undirected graph with the vertex set $V(G)$. If G contains n vertices then it is said to be an n-vertex graph and we write $|V(G)| = n$. Two graphs G and H are isomorphic if there exists a one-to-one correspondence between their vertex sets which preserves adjacency. A sub-graph of G is a graph having all of its vertices and edges in G. The complete graph is a graph in which any two distinct vertices are adjacent.

Throughout this paper all rings are commutative with non-zero identity and all modules unitary. We associate a graph $G_R(M)$ to an R-module M whose vertices are elements of M in these way that two distinct vertices x and y are adjacent if and only if there exists r belong to R that $x = ry$ or $y = rx$. We investigate the relationship between the algebraic properties of an R-module M and the properties of the associated graph $G_R(M)$ namely Scalar-product graph of M.

Let $G = (V;E)$ be a graph. We say that G is connected if there is a path between any two distinct vertices of G. For vertices x and y of G, we define $d(x;y)$ to be the length of a shortest path from x to y ($d(x;x) = 0$ and $d(x;y) = \infty$ if there is no such path). The diameter of G is $diam(G) = sup\{d(x;\ y): x;\ y \in V(G)\}$. The girth of a graph G, denoted by $gr(G)$, is the length of the shortest cycle in G. A graph with no cycle has infinite girth. For a vertex $v \in G$, neighbours of v denotes $N(v)$ is equal $\{u \in V(G) \setminus$

$\{v\}|\ v\ is\ adjacent\ to\ u\ \}$. In a graph G, a set $S \subseteq V(G)$ is an independent set if the sub-graph induced by S contains no edge. The independence number $\alpha(G)$ is the maximum size of an independent set in G.

In next section, we compute diameter and girth of $G_R(M)$ and discuss planarity of $G_R(M)$. In last section we show $G_R(M)$ is weakly perfect.

2 DIAMETER AND GIRTH OF $G_R(M)$

Definition 2.1. Let M be R-module and $x \in M$, we denote set of vertices that is adjacent to x in $G_R(M)$ by $T_x(M) = \{m \in M : rm = x \text{ for some } r \in R\}$. The torsion element of M is $T_0(M)$. The $T_x(M)$ is set of neighbors of x or N(x). Note that if $G_R(M)$ be a Scalar product graph of R-module M, then $x, y \in M$ is adjacent if and only if $x \in T_y(M)$ or $y \in T_x(M)$.

Remark 2.2. Let M be a finite R-module and $G_R(M)$ be a Scalar product graph of M. If M is torsion then for every $m \in M$, we have $m{\sim}0$ and $\deg(0) = |M| - 1$ also $\operatorname{diam}(G_R(M)) \leq 2$. Also, If M is torsion-free then vertex 0 is isolated vertex.

Example 2.3.

(1) Let M be a free R-module, then one can see that M is torsion-free, thus 0 is isolated vertex. Also, if V is vector space over field K then V is torsion-free, therefore 0 is isolated vertex.

(2) $\mathbb{Q}$ is torsion-free $\mathbb{Z}$-module. Therefore 0 is isolated vertex.

(3) If R is a integral domain and Q its field of fractions, then $\frac{Q}{R}$ is a torsion R-module. Therefore $diam(G_R(\frac{Q}{R})) \leq 2$.

(4) Consider a linear operator L acting on a finite-dimensional vector space V. If we view V as an $F[L]$-module in the natural way, then, V is a torsion $F[L]$-module. Then $T(V) = V$ as a result by previous remark we have $\deg(0) = |V| - 1$ and $diam(G_{F[L]}(V)) \leq 2$.

Remark 2.4. Let $G_R(M)$ be a Scalar product graph of R-module M. If $x, y \in M$ then x is adjacent to y if and only if $< x > \subseteq < y >$ or $< y > \subseteq < x >$ in the other words $Rx \subseteq Ry$ or $Ry \subseteq Rx$.

Lemma 2.5. Let M be an R-module and $x, y \in M$. If $< x > = < y >$, then x is adjacent to y in $G_R(M)$ and for all $z \in M$, x is adjacent to z if and only if y is adjacent to z.

Proof. Suppose $< x > = < y >$ then $< x > \subseteq < y >$. So x is adjacent to y. If z is adjacent to x, then $< z > \subseteq < x >$ or $< x > \subseteq < z >$. Hence $< z > \subseteq < y >$ or $< y > \subseteq < z >$. So z is adjacent to y. Similarly, if y is adjacent to z then x is adjacent to z. This concludes that any two vertices that generate the same sub-modules will have exactly the same set of neighbors.

Corollary 2.6. Let M be an R-module and $x, y \in M$. The cyclic sub-modules Rx, Ry are maximal, Then x is not adjacent to y in $G_R(M)$.

Proof. Suppose x is adjacent to y in $G_R(M)$. Without loss of generality suppose that $Rx \subseteq Ry$ which is contradiction by maximality of Rx.

Lemma 2.8. Let M be a R-module. Then scalar product graph $G_R(M)$ is complete if and only if the cyclic sub-modules of M are linearly ordered by inclusion relation.

Proof. Let M be a R-module and $N_1 = < a >$;$N_2 = < b >$ be two cyclic sub-modules of M that $a \neq b$ in M. Since scalar product graph $G_R(M)$ is complete then a and b is adjacent. We have $< a > \subseteq < b >$ nor $< b > \subseteq < a >$ and $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. Therefore cyclic submodules of M be linearly ordered by inclusion relation. Conversely, Let M be R-module which linearly ordered cyclic sub-modules by inclusion relation. If $a \neq b$ is two vertices of $G_R(M)$ then $< a > \subseteq < b >$ nor $< b > \subseteq < a >$. Therefore we have a and b are adjacent in $G_R(M)$. Therefore $G_R(M)$ is complete.

Theorem 2.9. Let $\mathbb{Z}_n$ as $\mathbb{Z}$-module. If p, m is prime and positive integer number, Then for $n = 1, p, p^m$, scalar product graph $G_{\mathbb{Z}}(\mathbb{Z}_n)$ is complete.

Proof. If $n = 1$, then $G_{\mathbb{Z}}(\mathbb{Z}_n)$ is a complete graph with one vertex. Let $n = p^m$ where p is a prime number and $m \in N$ and $\mathbb{Z}_{p^m}$ be a finite cyclic $\mathbb{Z}$-module. Then for every divisor d of n, there exists a unique sub-module of order d and all these (cyclic) sub-module of $M = \mathbb{Z}_{p^m}$ are precisely the following:

$M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_m (= M)$, where $|M_i| = p^i$. Thus by lemma 2.8, we have $G_{\mathbb{Z}}(\mathbb{Z}_n)$ is complete.

Corollary 2.10. Let R be a ring and M a finite R-module. If $G_R(M)$ is complete then M is a cyclic R-module.

Proof. Let M be a finite R-module and $G_R(M)$ be complete graph. If M is not a cyclic R-module, then M has at least two elements a, b such that $a \notin < b >$ and $b \notin < a >$ which imply a and b are not adjacent in $G_R(M)$ contradicting the fact that $G_R(M)$ is complete.

Example 2.7. Let $M = \mathbb{Z}_6$ be $\mathbb{Z}$-module. Scalar product $G_{\mathbb{Z}}(\mathbb{Z}_6)$ have shown in Figure 1.

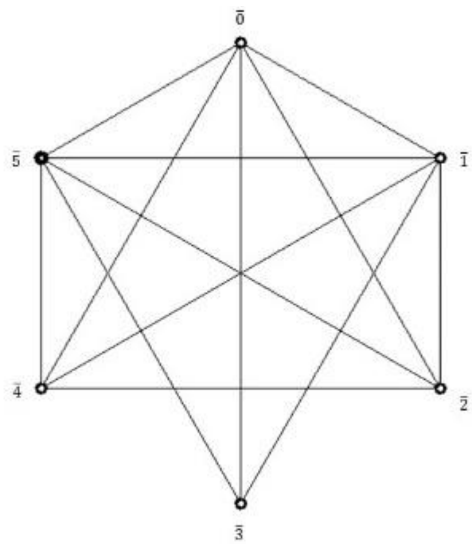

Figure 1. Scalar Product of $\mathbb{Z}$-module $\mathbb{Z}_6$

Authors in [1] introduced the cozero-divisor graph of a commutative ring R denoted by $\Gamma'(R)$ as a graph with vertices $W(R)^* = W(R) \setminus \{0\}$ where $W(R)$ is the set of all non-unit elements of R and two distinct vertices x and y are adjacent if and only if $x \notin Ry$ and $y \notin Rx$ where Rc is a ideal generated by $c \in R$. Let M be an R-module and $W_R(M) = \{x \in M \,|Rm \neq M\}$. If we suppose R as R-module $W_R(M)$ is set of all non-units elements of R. In [2] authors investigate cozero-divisor graphs on R-module M which vertices from $W_R(M)^* = W_R(M) \setminus \{0\}$ and two distinct vertices m and n are adjacent if and only if $m \notin Rn$ and $n \notin Rm$, and they studied girth, independent number, clique number and planarity of this graph.

If M be an R-module, the subgraph of $G_R(M)$ which vertices are $W_R(M)^*$ is complement of cozero-divisors graph of M. Then we denote $G_R(M) = \Gamma_1 \vee \Gamma_1$ which Γ_1 is a complete graph with $|W_R(M)^*|$ vertices and Γ_2 is complement of cozero-divisor graph of M.

One graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph G is a graph resulting from the subdivision of edges in G. The subdivision of some edge e with end-points $\{u, v\}$ yields a graph containing one new vertex w, and with an edge set replacing e by two new edges, $\{u, w\}$ and $\{w, v]$. Kuratowski Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$, where K_n is a complete graph with n vertices and $K_{m,n}$ is a complete bipartite graph, for positive integers m, n.

Proposition 2.11. Let M a finite R-module. If $|W_R(M)^*| \geq 5$, then $G_R(M)$ is not planar.

Proof. If $|W_R(M)^*| \geq 5$ then subgraph of $G_R(M)$ which vertices are$W_R(M)^*$ is complete. By Kuratowskis Theorem, we have $G_R(M)$ is not planar.

3 WEAKLY PERFECT

For a graph G, a k-colouring of the vertices of G is an assignment of k colors to the vertices of G in such a way that no two adjacent vertices receive the same color. The chromatic number of G, denoted by $\chi(G)$, is the smallest number k such that G admits a k-coloring. A clique of G is a complete sub-graph of G and the number of vertices in a largest clique of G, denoted by $\omega(G)$, is called the clique number of G. It is easy to see that $\chi(G) \geq \omega(G)$, because every vertex of a clique should get a different color. A graph G is called weakly perfect if $\chi(G) = \omega(G)$. If $M = \mathbb{Z}_n$ be an finite $\mathbb{Z}$-module, then $G_{\mathbb{Z}}(M)$ is weakly perfect.

Example 3.1. Chromatic number and clique number of $G_{\mathbb{Z}}(\mathbb{Z}_n)$ foe some n is listed in below (p is prime number):

n	$\chi(G_{\mathbb{Z}}(\mathbb{Z}_n))$	$\omega(G_{\mathbb{Z}}(\mathbb{Z}_n))$
$n=1$	1	1
$n=p$	p	p
$n=p^n$	p^n	p^n
$n=2p$	$2p-1$	$2p-1$
$n=3p$	$3p-2$	$3p-2$

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