Iran University of Science &Technology

3TH INTERNATIONAL CONFERENCE ON COMBINATORICS, CRYPTOGRAPHY AND COMPUTATION

Proceedings of the 3nd International Conference on Combinatorics, Cryptography and Computation (I4C2018)

Perfect 2-colorings of inflation of k4

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ABSTRACT

Perfect coloring is a generalization of the notion of completely regular codes, given by Delsarte. A perfect m-coloring of a graph G with m colors is a partition of the vertex set of G into m parts A1, . . . , Am such that, for all i, j ∈ {1,…,m}, every vertex of Ai is adjacent to the same number of vertices, namely, aij vertices, of Aj. The matrix A = (aij) i;j ∈ {1,2,…,m}, is called the parameter matrix. We study the perfect 2-colorings (also known as the equitable partitions into two parts) of the inflation of k4. In particular, we classify all the realizable parameter matrices of perfect 2-colorings for the inflation of k4.

KEYWORDS: ***perfect coloring, parameter matrices, cubic graph, inflated graph***

1 INTRODUCTION

The concept of a perfect m-coloring plays an important role in graph theory, algebraic combinatorics, and coding theory (completely regular codes). There is another term for this concept in the literature as "equitable partition" (see [11]).

The existence of completely regular codes in graphs is a historical problem in mathematics. Completely regular codes are a generalization of perfect codes. In 1973, Delsarte conjectured the non-existence of nontrivial perfect codes in Johnson graphs. Therefore, some effort has been done on enumerating the parameter matrices of some Johnson graphs, including J (6; 3), J (7; 3), J (8; 3), J (8; 4), and J (v; 3) (v odd) (see [4, 5, 9]). Fon-Der-Flass enumerated the parameter matrices (perfect 2-colorings) of n-dimensional hypercube Qn for n < 24. He also obtained some constructions and a necessary condition for the existence of perfect 2-colorings of the n-dimensional cube with a given parameter matrix (see [6, 7, 8]). In this article, we enumerate the parameter matrices of all perfect 2-colorings of the inf(k4).

Definition1.1 For a graph G and an integer m, a mapping T: V (G) $\rightarrow$ {1, 2,…, m} is called a perfect m-coloring with matrix A = (aij) i;j ∈ {1,2,…,m}, if it is surjective, and for all I, j, for every vertex of color i, the number of its neighbours of color j is equal to aij . The matrix A is called the parameter matrix of a perfect coloring. In the case m = 2, we call the first color white that show by W, the second color black that show by B. In this paper, we generally show a parameter matrix by

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Remark 1.2 In this paper, we consider all perfect 2-colorings, up to renaming the colors. We identify the perfect 2-coloring with the matrix

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$,

obtained by switching the colors with the original coloring.

The inflation GI of a graph G with n (G) vertices and m (G) edges is obtained from G by replacing every vertex of degree d of G by a clique Kd. in this paper, we consider to the graph inf(k4). The graph k4 and its inflation will be shown in figure1.

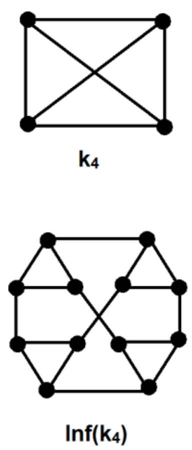

Figure 1: graph k4 and inf(k4)

2 PRELIMINARIES AND ANALYSIS

Now, we first give some results concerning necessary conditions for the existence of perfect 2-colorings of a k-regular graph with a given parameter matrix A= (aij) i;j=1;2.

The simplest condition for the existence of perfect 2-colorings of a k-regular graph with the matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is:

$a_{11} + a_{12} = a_{21} + a_{22} = 2$.

Also, when the graph is connected, it is clear that neither a_{11} nor a_{12} cannot be equal to zero, otherwise white and black vertices of the graph would not be adjacent, that is impossible, because the graph is connected. By the given conditions, we can see that a parameter matrix

A1=$\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$, A2=$\begin{bmatrix} 0 & 3 \\ 2 & 1 \end{bmatrix}$, A3=$\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$, A4=$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, A5=$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, A6 =$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

The next proposition gives a formula for calculating the number of white vertices in a perfect 2-coloring.

Proposition 2.1 ([4]) If W is the set of white vertices in a perfect 2-coloring of a graph G with matrix A = (aij)i;j=1;2, then

$$|W| = |V(G)| \frac{a_{12}}{a_{12}+a_{21}}.$$

The number θ is called an eigenvalue of a graph G, if θ is an eigenvalue of the adjacency matrix of this graph. The number θ is called an eigenvalue of a perfect coloring T into three colors with the matrix A, if θ is an eigenvalue of A. The following theorem demonstrates the connection between the introduced notions.

Theorem 2.2 [6] If T is a perfect coloring of a graph G in m colors, then any eigenvalue of T is an eigenvalue of G.

The next theorem can be useful to find the eigenvalues of a parameter matrix.

Corollary 2.3 It is easy to see that every perfect 2-coloring of a k-regular graph with parameter matrix $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ has two eigenvalues: one is k, and the other is a-c such that we obviously have a-c$\neq$ k. So, from Theorem 2:4, we conclude that a-c is an eigenvalue of a k-regular connected graph which is not equal to k.

The eigenvalues of the Haywood graph are stated in the next theorem.

Theorem 2.4 [12] The distinct eigenvalues of the graph inf(k4) are the numbers 3, $2, 0, -1, -2$.

3 PERFECT 3-COLORINGS OF GRAPH INF(K_4)

In this section, we enumerate the parameter matrices of all perfect 2-colorings of the cubic connected graphs of order less than 10. As it shown in the Section 2, only the matrices

A1; A2; …; A6 can be parameter ones. By Proposition 2:3, it is clear that the matrix A2 cannot be a parameter matrix.

By using the Theorems 2.2 and 2.4 and Proposition 2.1, it can be seen that only the following matrices can be parameter ones.

A1 $=\begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$, A3 $=\begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$, A4 $=\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, A5 $=\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$.

We label the vertices of the graph inf(k4) by a1, a2,…,a12 as follows.

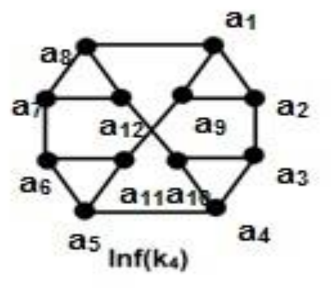

Figure 2: graph inf(k4)

Theorem 3.1 There are no perfect 2-colorings with the matrix A1 for the graph inf(k4).

Proof: Contrary to our claim, suppose that T is a perfect 2-coloring with the matrix A1 for graph inf(k4). According to the matrix A1, each vertex with color 1 has one adjacent with color 2. If T (a1) = 1, then we have T (a2) = T (a8) = T (a9) = 2,

which is a contradiction with the second row of matrix A1. If T (a1) = 2, then we have T (a2) = T (a8) = T (a9) =1, which is a contradiction with the first row of matrix A1. Therefore the graph inf(k4) has no perfect 2-coloring with matrix A1.

Theorem 3.2 The parameter matrices of cubic graphs of order 10 are listed in the table 1.

Table1:

Graph	MatixA1	MatixA2	MatixA3	MatixA4	MatixA5	MatixA6
Inf(k4)	×	×	√	√	√	×

Proof: As it has been shown in the previous section, only matrices A1, A3, A4, and A5 can be a parameter matrix. By using Theorem 3.1 we can see the graph inf(k4) has no perfect 2-coloring with the matrix A_1. With consideration of cubic graphs eigenvalues and using Proposition 2.1 and Theorem 2.4, it can be seen that the graph inf(k_4) can have perfect 2-coloring with matrices A_2, A_3, A_4 and A_5.

Now we introduce the mappings graph inf(k_4) that have perfect 2-clorings with the parameter matrices. The graph inf(k4) has perfect 2-colorings with the matrices A_3, A_4 and A_5. Consider the mapping T_1, T_2 and T_3 as follows:

$T_1(a_1) = T_1(a_6) = T_1(a_{10}) = 1$, $T_1(a_2) = T_1(a_3) = T_1(a_4) = T_1(a_5) = T_1(a_7) = T_1(a_8) = T_1(a_9) = T_1(a_{11}) = T_1(a_{12}) = 2$,

$T_2(a_1) = T_2(a_4) = T_2(a_6) = T_2(a_7) = T_2(a_9) = T_2(a_{10}) = 1$, $T_2(a_2) = T_2(a_3) = T_2(a_5) = T_2(a_8) = T_2(a_{11}) = T_2(a_{12}) = 2$,

$T_3(a_9) = T_3(a_{10}) = T_3(a_{11}) = T_3(a_{12}) = 1$, $T_3(a_1) = T_3(a_2) = T_3(a_3) = T_3(a_4) = T_3(a_5) = T_3(a_6) = T_3(a_7) = T_3(a_8) = 2$.

It is clear that T_1, $T_2$and T_3 are perfect 2-coloring with the matrices A_3, A_4 and A_5, respectively.

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