On Some Properties and Identities of Van Der Laan Hybrinomail Sequence

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ABSTRACT

Hybrid numbers are the generalization of complex, hyperbolic and dual numbers. In this paper the Van Der Laan Hybrinomial sequence is introduced. We give Binet-Like formula, Partial sum, generating function and exponential generating function of this kind of polynomial sequence. In addition we present other properties and identities about Van Der Laan Hybrinomial sequence.

KEYWORDS: Van Der Laan sequence, Hybrinomial sequence, identities, generating function.

1 INTRODUCTION

Various researchers have studied different kinds of recursion sequences such as Pell sequence, Pell Lucas sequence, Padovan and Perrin sequences and Jacobsthal sequence. They established new results about these sequences. Ozdemir[8] introduced the hybrid numbers as a generalization of complex hyperbolic and dual numbers. He has stated that the set H of hybrid numbers Z is of the form

$$H = \{Z = a + bi + c\epsilon + dh; a, b, c, d \in \mathbb{R}\},$$

where i, ϵ, h are operators such that

$$i^2 = -1\,, \epsilon^2 = 0, \qquad ih = -hi = \epsilon + i.$$

For more results about the hybrid number we refer to [8]. The conjugate of hybrid number Z is defined by

$$\overline{Z} = \overline{a + bi + c\epsilon + dh} = a - bi - c\epsilon - dh.$$

The real number $R(Z) = Z\overline{Z} = \overline{Z}Z = a^2 + b^2 - 2bc - d^2$ is called the character of the hybrid number Z. Liana and Wloch [7] introduced the Jacobsthal and Jacobsthal Lucas hybrid numbers and investigated some of their properties. The authors in [11] introduced Van Der Laan Hybrid numbers and presented some results about this kind of hybrid numbers. In this paper we introduce the Van Der Laan Hybrinomial sequence. We obtain Binet-like formula, partial sum, generating function, exponential generating function, character and norm of this sequence. In addition we present other properties and identities about Van Der Laan Hybrinomial sequence.

For more information about Van der Laan sequence, Pell sequence, Pell-Lucas sequence, Jacobsthal-Lucas sequence, Padovan and Perrin sequences and the other related number sequences see [1]- [6], [8],[10-15].

2 VAN DER LAAN SEQUENCE

The Van Der Laan [5] sequence $(V_n)$is defined by the recursion relation

$$V_n = V_{n-2} + V_{n-3} \text{ for all } n \geq 3,$$

With initial values $V_0 = 0$, $V_1 = 1, V_2 = 0$. The first values of (V_n) are

$$0,1,0,1,1,1,2,2,3,4,5,7,9,12,16,21,28,37,49,65,86,114$$

Also the Binet-like-formula [5] for the Van Der Laan sequences is:

$$V_n = \frac{{r_1}^n}{(r_1 - r_2)(r_1 - r_3)} + \frac{{r_2}^n}{(r_2 - r_1)(r_2 - r_3)} + \frac{{r_3}^n}{(r_1 - r_3)(r_2 - r_3)},$$

where r_1, r_2, r_3 are the roots of the equation $x^3 - x - 1 = 0$. Also from [2] we have $r_1 + r_2 + r_3 = 0, r_1 r_2 r_3 = 1, r_1 r_2 + r_2 r_3 + r_1 r_3 = -1$.

3 VAN DER LAAN HYBRINOMIAL SEQUENCE

Definition (3.1). The Van Der Laan Hybrinomial sequence $\left\{V_n^{[H]}(x)\right\}$ is defined by the relation

$$V_n^{[H]}(x) = V_n(x) + V_{n+1}(x)i + V_{n+2}(x)\epsilon + V_{n+3}(x)h, \quad (3.1)$$

where $(V_n(x))$ is the Van Der Laan Polynomial sequence and is defined by

$$V_n(x) = \begin{cases} 0 & n = 0,2 \\ 1 & n = 1,3,4 \\ xV_{n-2}(x) + V_{n-3}(x) & n \geq 5 \end{cases}$$

From the above definitions, the first few values of Van Der Laan Polynomial sequence are $V_0(x) = 0$, $V_1(x) = 1$, $V_2(x) = 0$, $V_3(x) = 1$, $V_4(x) = 1, V_5(x) = x$, $V_6(x) = x + 1$, $V_7(x) = x^2 + 1$.

The characteristic equation of Van Der Laan Polynomial sequence $V_n(x)$ is $t^3 - xt - 1 = 0$. For each real value of x this cubic equation has three distinct roots α, β, γ. The initial value of the Van Der Laan Hybrinomial sequence are

$V_0^{[H]}(x) = V_0(x) + V_1(x)i + V_2(x)\epsilon + V_3(x)h = i + h,$

$V_1^{[H]}(x) = V_1(x) + V_2(x)i + V_3(x)\epsilon + V_4(x)h = 1 + \epsilon + h,$

$V_2^{[H]}(x) = V_2(x) + V_3(x)i + V_4(x)\epsilon + V_5(x)h = i + \epsilon + x,$

$V_3^{[H]}(x) = V_3(x) + V_4(x)i + V_5(x)\epsilon + V_6(x)h = 1 + 1i + x\epsilon + (x + 1)h,$

$V_4^{[H]}(x) = V_4(x) + V_5(x)i + V_6(x)\epsilon + V_7(x)h = 1 + xi + (x + 1)\epsilon + (x^2 + 1)h.$

By definition of the Van Der Laan Hybrinomial sequence and the character of a hybrid numbers we get

$$R\left(V_n^{[H]}(x)\right) = {V_n}^2(x) + {V_{n+1}}^2(x) - 2V_{n+1}(x)V_{n+2}(x) - {V_{n+3}}^2(x). \blacksquare$$

Now we have the following theorem about the norm of van Der Laan Hybrinomial sequence$\left\{V_n^{[H]}(x)\right\}$.

Theorem (3.2). The norm of the Van Der Laan hybrinomial sequence is given by

$$\left\|V_n^{[H]}(x)\right\|^2 = 2V_{n+1}(\mathrm{x})[V_{n+2}(\mathrm{x}) + V_n(\mathrm{x})].$$

Proof. We have

$R\left(V_n^{[H]}(x)\right) = {V_n}^2(x) + {V_{n+1}}^2(x) - 2V_{n+1}(x)V_{n+2}(x) - {V_{n+3}}^2(x).$

So we get

$R\left(V_n^{[H]}(x)\right) = {V_n}^2(x) + {V_{n+1}}^2(x) - 2V_{n+1}(x)V_{n+2}(x) - \left(V_n(x) + V_{n+1}(x)\right)^2 =$
${V_n}^2(x) + {V_{n+1}}^2(x) - 2V_{n+1}(x)V_{n+2}(x) - {V_n}^2(x) - {V_{n+1}}^2(x) - 2V_n(x)V_{n+1}(x) =$
$-2V_{n+1}(x)(V_{n+2}(x) + V_n(x)).$

Hence we have

$$\left\|V_n^{[H]}(x)\right\| = \sqrt{\left|R\left(V_n^{[H]}(x)\right)\right|} = \sqrt{\left|{V_n}^2(\mathrm{x}) + {V_{n+1}}^2(\mathrm{x}) - 2V_{n+1}(x)V_{n+2}(\mathrm{x}) - {V_{n+3}}^2(x)\right|}.$$

Therefore we get

$$\begin{aligned}\left\|V_n^{[H]}(x)\right\|^2 &= \left|{V_n}^2(\mathrm{x}) + {V_{n+1}}^2(\mathrm{x}) - 2V_{n+1}(x)V_{n+2}(\mathrm{x}) - (V_n(\mathrm{x}) + V_{n+1}(\mathrm{x}))^2\right| \\ &= \left|{V_n}^2(\mathrm{x}) + {V_{n+1}}^2(\mathrm{x}) - 2V_{n+1}(x)V_{n+2}(\mathrm{x}) - {V_n}^2(\mathrm{x}) - {V_{n+1}}^2(\mathrm{x}) \right. \\ &\quad \left. - 2V_n(x)V_{n+1}(x)\right| = |-2V_{n+1}(\mathrm{x})(V_{n+2}(\mathrm{x}) + V_n(\mathrm{x}))| \\ &= 2V_{n+1}(\mathrm{x})(V_{n+2}(\mathrm{x}) + V_n(\mathrm{x})).\end{aligned}$$

Thus the proof is completed.

$\blacksquare$

Ozemir [8] defined the matrix representation of hybrid numbers by

$$M_{a+bi+c\epsilon+dh} = a\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + b\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} + c\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix} + d\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}.$$

Now we display the matrix representation of the Van Der Laan Hybrinomial sequence.

Remark (3.3). The matrix representation of Van Der Laan Hybrinomial sequence is

$$M_{V_r^{[H]}} = \begin{bmatrix}V_r(x) + V_{r+2}(x) & V_{r+1}(x) - V_{r+2}(x) + V_{r+3}(x)\\ -V_{r+1}(x) + V_{r+2}(x) + V_{r+3}(x) & V_r(x) - V_{r+2}(x)\end{bmatrix}.$$

Proof. From the definition of Van Der Laan Hybrinomial sequence we have

$$V_r^{[H]}(x) = V_r(x) + V_{r+1}(x)i + V_{r+2}(x)\epsilon + V_{r+3}(x)h.$$

Consequently by the matrix representation of hybrid numbers introduced by Ozdemir we get

$$M_{V_r^{[H]}(x)} = V_r(x)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + V_{r+1}(x)\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} + V_{r+2}(x)\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix} + V_{r+3}(x)\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

$$= \begin{bmatrix} V_r(x) + V_{r+2}(x) & V_{r+1}(x) - V_{r+2}(x) + V_{r+3}(x) \\ -V_{r+1}(x) + V_{r+2}(x) + V_{r+3}(x) & V_r(x) - V_{r+2}(x) \end{bmatrix}.$$

■

Corollary (3.4). Determinant for the matrix representation of Van Der Laan Hybrinomial sequence is $-2V_{r+1}(x)\big(V_{r+2}(x) + V_r(x)\big)$ and we have

$$\left|\det\left(M_{V_r^{[H]}(x)}\right)\right| = 2V_{r+1}(x)\big(V_{r+2}(x) + V_r(x)\big).$$

Proof. From Remark (3.3) we know that

$$M_{V_r^{[H]}(x)} = \begin{bmatrix} V_r(x) + V_{r+2}(x) & V_{r+1}(x) - V_{r+2}(x) + V_{r+3}(x) \\ -V_{r+1}(x) + V_{r+2}(x) + V_{r+3}(x) & V_r(x) - V_{r+2}(x) \end{bmatrix}$$

Hence we have

$$\det\left(M_{V_r^{[H]}(x)}\right) = \det\left(\begin{bmatrix} V_r(x) + V_{r+2}(x) & V_{r+1}(x) - V_{r+2}(x) + V_{r+3}(x) \\ -V_{r+1}(x) + V_{r+2}(x) + V_{r+3}(x) & V_r(x) - V_{r+2}(x) \end{bmatrix}\right)$$
$$= \Big(\big(V_r(x) + V_{r+2}(x)\big)(V_r(x) - V_{r+2}(x))\Big)$$
$$- \Big((V_{r+1}(x) - V_{r+2}(x) + V_{r+3}(x))(-V_{r+1}(x) + V_{r+2}(x) + V_{r+3}(x))\Big).$$

By some computations we have

$$\det\left(M_{V_r^{[H]}(x)}\right) = V_r{}^2(\mathrm{x}) + V_{r+1}{}^2(\mathrm{x}) - 2V_{r+1}(x)V_{r+2}(\mathrm{x}) - V_r{}^2(\mathrm{x}) - V_{r+1}{}^2(\mathrm{x}) - 2V_r(x)V_{r+1}(x)$$
$$= -2V_{r+1}(x)(V_{r+2}(\mathrm{x}) + V_r(x)) = R\left(V_r^{[H]}(x)\right).$$

Therefore we have

$$\left|\det\left(M_{V_r^{[H]}}\right)\right| = 2V_{r+1}(V_{r+2} + V_r)\left|\det\left(M_{V_r^{[H]}(x)}\right)\right| = 2V_{r+1}(x)\big(V_{r+2}(x) + V_r(x)\big).$$

■

Theorem (3.5). The generating function for Van Der Laan polynomial sequence $\{V_n(x)\}$ is

$$f(t) = \sum_{n=0}^{\infty} V_n(x)t^n = \frac{t}{1 - t^2x - t^3}.$$

Proof. Suppose that the generating function of the Van Der Laan polynomial sequence $\{V_n(x)\}$ has the form

$$f(t) = \sum_{n=0}^{\infty} V_n(x)\, t^n = V_0(x) + V_1(x)t + V_2(x)t^2 + V_3(x)t^3 + \cdots \quad .$$

Then we have

$$t^2xf(t) = xV_0(x)t^2 + xV_1(x)t^3 + xV_2(x)t^4 + xV_3(x)t^5 + \cdots \quad ,$$

And

$$t^3f(t) = V_0(x)t^3 + V_1(x)t^4 + V_2(x)t^5 + V_3(x)t^6 + \cdots .$$

Thus we obtain

$f(t) - t^2xf(x) - t^3f(t) = (V_0(x) + V_1(x)t + V_2(x)t^2 + V_3(x)t^3 + \cdots) - (xV_0(x)t^2 + xV_1(x)t^3 + xV_2(x)t^4 + xV_3(x)t^5 + \cdots) - (V_0(x)t^3 + V_1(x)t^4 + V_2(x)t^5 + V_3(x)t^6 + \cdots)$

$= V_0(x) + V_1(x)t + (V_2(x) - xV_0(x))t^2 + (V_3(x) - xV_1(x) - V_0(x))t^2 + \cdots + (V_n(x) - xV_{n-2}(x) - V_{n-3}(x))t^n + \cdots$.

By Definition of Van Der Laan Polynomial sequence we have $V_n(x) - xV_{n-2}(x) - V_{n-3}(x) = 0$. Also we know that $V_0(x) = 0$, $V_1(x) = 1$ and $V_1(x) = 1$. Hence we obtain

$$f(t) - t^2xf(t) - t^3f(t) = V_0(x) + V_1(x)t + \big(V_2(x) - xV_0(x)\big)t^2 = t.$$

Thus we get

$$f(t)(1-t^2x-t^3)=t.$$

Consequently we have

$$f(t)=\sum_{n=0}^{\infty} V_n(x)t^n=\frac{t}{1-t^2x-t^3}. \blacksquare$$

Lemma (3.6). Let $n \geq 0$ be an integer. The Binet-Like formula for the Van Der Laan Polynomial sequence $V_n(x)$ is

$$V_n(x)=\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} .$$

Where α, β, γ are the roots of the characteristic equation $t^3-xt-1=0$.

Proof. We know that the recursive relation $V_n(x)=xV_{n-2}(x)+V_{n-3}(x)$ has the characteristic equation$g(t)=t^3-xt-1=0$. For an arbitrary value of x we know that this equation has three distinct roots α, β, γ. Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are the roots of $h(t)=g\left(\frac{1}{t}\right)=1-t^2x-t^3=0$.

In exact we have

$$h(t)=1-t^2x-t^3=(1-\alpha t)(1-\beta t)(1-\gamma t).$$

According to the generating function of Van der Laan polynomial sequence $V_m(x)$ we have

$$\begin{aligned} f(t)&=\frac{t}{1-t^2x-t^3}=\frac{A}{1-\alpha t}+\frac{B}{1-\beta t}+\frac{C}{1-\gamma t} \\ &=A\sum_{n=0}^{\infty}(\alpha t)^n+B\sum_{n=0}^{\infty}(\beta t)^n+C\sum_{n=0}^{\infty}(\gamma t)^n . \qquad (3.6.1) \end{aligned}$$

Thus we have

$$g(t)=\frac{t}{1-xt-t^3}=\frac{A(1-\beta t)(1-\gamma t)+B(1-\alpha t)(1-\gamma t)+C(1-\alpha t)(1-\beta t)}{(1-\alpha t)(1-\beta t)(1-\gamma t)}.$$

Therefore by comparison of the left and right sides of this equality we get that

$$t=A(1-\beta t)(1-\gamma t)+B(1-\alpha t)(1-\gamma t)+C(1-\alpha t)(1-\beta t).$$

If we substitute t by $\frac{1}{\alpha}$ we find that

$$\frac{1}{\alpha}=A\left(1-\frac{\beta}{\alpha}\right)\left(1-\frac{\gamma}{\alpha}\right).$$

Hence we get

$$\alpha=A(\alpha-\beta)(\alpha-\gamma)$$

Consequently we obtain

$$A=\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}.$$

Similarly we have

$$B=\frac{\beta}{(\beta-\alpha)(\beta-\gamma)} \quad ,C=\frac{\gamma}{(\gamma-\alpha)(\gamma-\beta)} .$$

Thus by (3.6.1) we get

$$G(t)=\sum_{n=0}^{\infty}\frac{\alpha\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}t^n+\sum_{n=0}^{\infty}\frac{\beta\beta^n}{(\beta-\alpha)(\beta-\gamma)}t^n+\sum_{n=0}^{\infty}\frac{\gamma\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}t^n$$

$$= \sum_{n=0}^{\infty} \left[\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \right] t^n.$$

Consequently we obtain

$$V_n(x) = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}$$

Thus the proof is completed.

Theorem (3.7). Let $n \geq 0$ be an integer. Then the Binet-Like formula for Van Der Laan Hybrinomial sequence $V_n^{[H]}(x)$ is

$$V_n^{[H]}(x) = \left(\frac{1+\alpha i+\alpha^2\epsilon+\alpha^3 h}{(\alpha-\beta)(\alpha-\gamma)}\right)\alpha^{n+1} + \left(\frac{1+\beta i+\beta^2\epsilon+\beta^3 h}{(\beta-\alpha)(\beta-\gamma)}\right)\beta^{n+1} + \left(\frac{1+\gamma i+\gamma^2\epsilon+\gamma^3 h}{(\gamma-\alpha)(\gamma-\beta)}\right)\gamma^{n+1}.$$

Where α, β, γ are the roots of the equation $t^3 - xt - 1 = 0$.

Proof. From Binet-like-formula of Van Der Laan polynomial sequence we have

$$V_n(x) = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} ,$$

Where α, β, γ are the roots of the equation $t^3 - xt - 1 = 0$.

According to the definition of Van Der Laan Hybrinomial sequence $V_r^{[H]}(x)$ we have
$V_n^{[H]}(x) = V_n(x) + V_{n+1}(x)i + V_{n+2}(x)\epsilon + V_{n+3}(x)h.$

Thus we have

$$\begin{aligned} V_r^{[H]}(x) &= \left(\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \right) \\ &+ \left(\frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)} \right) i \\ &+ \left(\frac{\alpha^{n+3}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+3}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+3}}{(\gamma-\alpha)(\gamma-\beta)} \right) \epsilon \\ &+ \left(\frac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)} \right) h \\ &= \left(\frac{1+\alpha i+\alpha^2\epsilon+\alpha^3 h}{(\alpha-\beta)(\alpha-\gamma)}\right)\alpha^{n+1} + \left(\frac{1+\beta i+\beta^2\epsilon+\beta^3 h}{(\beta-\alpha)(\beta-\gamma)}\right)\beta^{n+1} \\ &+ \left(\frac{1+\gamma i+\gamma^2\epsilon+\gamma^3 h}{(\gamma-\alpha)(\gamma-\beta)}\right)\gamma^{n+1}. \ \blacksquare \end{aligned}$$

Lemma (3.8). Let $V_k(x)$ is the Van Der Laan polynomial sequence and $n \geq 0$ be an integer. Then

$$\sum_{k=0}^{n} V_k(x) = \frac{1}{x}[V_{n+2}(x) + V_{n+1}(x) + V_n(x) - 1].$$

Proof. From the definition of Van Der Laan polynomial sequence we now that $V_n(x) = \frac{1}{x}(V_{n+2}(x) - V_{n-1}(x))$. Thus we have

$$V_1(x) = \frac{1}{x}(V_3(x) - V_0(x)),$$
$$V_2(x) = \frac{1}{x}(V_4(x) - V_1(x)),$$
$$V_3(x) = \frac{1}{x}(V_5(x) - V_2(x)),$$
$$V_4(x) = \frac{1}{x}(V_6(x) - V_3(x)),$$
$$\vdots$$
$$V_{n-3}(x) = \frac{1}{x}(V_{n-1}(x) - V_{n-4}(x))$$
$$V_{n-2}(x) = \frac{1}{x}(V_n(x) - V_{n-3}(x)),$$
$$V_{n-1}(x) = \frac{1}{x}(V_{n+1}(x) - N_{n-2}(x)),$$
$$V_n(x) = \frac{1}{x}(V_{n+2}(x) - N_{n-1}(x)),$$

Therefore we get

$$\sum_{k=0}^{n} V_k(x) - V_0(x) = \frac{1}{x}(V_{n+2}(x) + V_{n+1}(x) + V_n(x) - V_0(x) - V_1(x) - V_2(x)).$$

Consequently we obtain

$$\sum_{k=0}^{n} V_k(x) = \frac{1}{x}[V_{n+2}(x) + V_{n+1}(x) + V_n(x) - 1].$$

Theorem (3.9). Let $n \geq 0$ be an integer. Then

$$\sum_{k=0}^{n} V_k^{[H]}(x) = \frac{1}{x}\left(\left[V_{n+2}^{[H]}(x) + V_{n+1}^{[H]}(x) + V_n^{[H]}(x)\right] - [1 + i + \epsilon + h]\right).$$

Proof. As we know $\sum_{k=0}^{n} V_k^{[H]}(x) = V_0^{[H]}(x) + V_1^{[H]}(x) + V_2^{[H]}(x) + \cdots + V_n^{[H]}(x)$. Thus we obtain

$$\begin{aligned}
\sum_{k=0}^{n} V_k^{[H]}(x) &= [V_0(x) + V_1(x)i + V_2(x)\epsilon + V_3(x)h] + [V_1(x) + V_2(x)i + V_3(x)\epsilon + V_4(x)h] + \cdots \\
&\quad + [V_n(x) + V_{n+1}(x)i + V_{n+2}(x)\epsilon + V_{n+3}(x)h] \\
&= [V_0(x) + V_1(x) + V_2(x) + \cdots + V_n(x)] \\
&\quad + [V_1(x) + V_2(x) + \cdots + V_{n+1}(x) + V_0(x) - V_0(x)]i \\
&\quad + [V_2(x) + V_3(x) + \cdots + V_{n+2}(x) + V_0(x) + V_1(x) - V_0(x) - V_1(x)]\epsilon \\
&\quad + [V_3(x) + V_4(x) + \cdots + V_{n+3}(x) + V_0(x) + V_1(x) + V_2(x) - V_0(x) - V_1(x) \\
&\quad - V_2(x)h] \\
&= \left[\sum_{k=0}^{n} V_k(x)\right] + \left[\sum_{k=0}^{n+1} V_k(x) - 0\right]i + \left[\sum_{k=0}^{n+2} V_k(x) - 1\right]\epsilon + \left[\sum_{k=0}^{n+3} V_k(x) - 1\right]h.
\end{aligned}$$

Therefore by lemma (3.8) we get:

$$\sum_{k=0}^{n} V_k^{[H]}(x)$$
$$= \frac{1}{x}[V_{n+2}(x) + V_{n+1}(x) + V_n(x) - 1] + \left(\frac{1}{x}[V_{n+3}(x) + V_{n+2}(x) + V_{n+1}(x) - 1]\right) i$$
$$+ \left(\frac{1}{x}[V_{n+4}(x) + V_{n+3}(x) + V_{n+2}(x) - 1]\right)\epsilon + \left(\frac{1}{x}[V_{n+5}(x) + V_{n+4}(x) + V_{n+3}(x) - 1]\right)h$$
$$= \frac{1}{x}[V_{n+2}(x) + V_{n+3}(x)i + V_{n+4}(x)\epsilon + V_{n+5}(x)h] + \frac{1}{x}[V_{n+1}(x) + V_{n+2}(x)i + V_{n+3}(x)\epsilon$$
$$+V_{n+4}(x)h] + \frac{1}{x}[V_n(x) + V_{n+1}(x)i + V_{n+2}(x)\epsilon + V_{n+3}(x)h] - \frac{1}{x}[1 + i + \epsilon + h]$$
$$= \frac{1}{x}\left(\left[V_{n+2}^{[H]}(x) + V_{n+1}^{[H]}(x) + V_n^{[H]}(x)\right] - [1 + i + \epsilon + h]\right).$$

Thus the proof is completed. ■

Theorem (3.10). Let $n \geq 0$ be an integer. Then

(a) $V_{n+1}^{[H]}(x) + V_n^{[H]}(x) = \left[(\alpha + 1)\left(\frac{1+\alpha i+\alpha^2\epsilon+\alpha^3 h}{(\alpha-\beta)(\alpha-\gamma)}\right)\right]\alpha^{n+1} + \left[(\beta + 1)\left(\frac{1+\beta i+\beta^2\epsilon+\beta^3 h}{(\beta-\alpha)(\beta-\gamma)}\right)\right]\beta^{n+1} + \left[(\gamma + 1)\left(\frac{1+\gamma i+\gamma^2\epsilon+\gamma^3 h}{(\gamma-\alpha)(\gamma-\beta)}\right)\right]\gamma^{n+1}$

(b) $V_{n+1}^{[H]}(x) - V_n^{[H]}(x) = \left[(\alpha - 1)\left(\frac{1+\alpha i+\alpha^2\epsilon+\alpha^3 h}{(\alpha-\beta)(\alpha-\gamma)}\right)\right]\alpha^{n+1} + \left[(\beta - 1)\left(\frac{1+\beta i+\beta^2\epsilon+\beta^3 h}{(\beta-\alpha)(\beta-\gamma)}\right)\right]\beta^{n+1} + \left[(\gamma - 1)\left(\frac{1+\gamma i+\gamma^2\epsilon+\gamma^3 h}{(\gamma-\alpha)(\gamma-\beta)}\right)\right]\gamma^{n+1}$,

Where α, β, γ are the roots of the equation $t^3 - xt - 1 = 0$.

Proof. We prove part **(a)**. Part (b) similarly is proved. By Binet-Like formula of Van Der Laan Hybrinomial sequence $V_n^{[H]}(x)$ we have

$$V_{n+1}^{[H]}(x) + V_n^{[H]}(x)$$
$$= \left[\left(\frac{1 + \alpha i + \alpha^2\epsilon + \alpha^3 h}{(\alpha - \beta)(\alpha - \gamma)}\right)\alpha^{n+2} + \left(\frac{1 + \beta i + \beta^2\epsilon + \beta^3 h}{(\beta - \alpha)(\beta - \gamma)}\right)\beta^{n+2}\right.$$
$$\left. + \left(\frac{1 + \gamma i + \gamma^2\epsilon + \gamma^3 h}{(\gamma - \alpha)(\gamma - \beta)}\right)\gamma^{n+2}\right]$$
$$+ \left[\left(\frac{1 + \alpha i + \alpha^2\epsilon + \alpha^3 h}{(\alpha - \beta)(\alpha - \gamma)}\right)\alpha^{n+1} + \left(\frac{1 + \beta i + \beta^2\epsilon + \beta^3 h}{(\beta - \alpha)(\beta - \gamma)}\right)\beta^{n+1}\right.$$
$$\left. + \left(\frac{1 + \gamma i + \gamma^2\epsilon + \gamma^3 h}{(\gamma - \alpha)(\gamma - \beta)}\right)\gamma^{n+1}\right]$$
$$= \left[(\alpha + 1)\left(\frac{1 + \alpha i + \alpha^2\epsilon + \alpha^3 h}{(\alpha - \beta)(\alpha - \gamma)}\right)\right]\alpha^{n+1} + \left[(\beta + 1)\left(\frac{1 + \beta i + \beta^2\epsilon + \beta^3 h}{(\beta - \alpha)(\beta - \gamma)}\right)\right]\beta^{n+1}$$
$$+ \left[(\gamma + 1)\left(\frac{1 + \gamma i + \gamma^2\epsilon + \gamma^3 h}{(\gamma - \alpha)(\gamma - \beta)}\right)\right]\gamma^{n+1}.$$

This proves the theorem. ■

Lemma (3.11). Let $n \geq 0$ be an integer. Then

$$V_n^{[H]}(x) - xV_{n-2}^{[H]}(x) - V_{n-3}^{[H]}(x) = 0.$$

Proof. By definition of Van Der Laan Hybrinomial sequence we have

$$\begin{aligned} &V_n^{[H]}(x) - xV_{n-2}^{[H]}(x) - V_{n-3}^{[H]}(x) \\ &\quad = [(V_n(x) + V_{n+1}(x)i + V_{n+2}(x)\epsilon + V_{n+3}(x)h)] \\ &\quad - x[(V_{n-2}(x) + V_{n-1}(x)i + V_n(x)\epsilon + V_{n+1}(x)h)] \\ &\quad - [(V_{n-3}(x) + V_{n-2}(x)i + V_{n-1}(x)\epsilon + V_n(x)h)] \\ &\quad = [V_n(x) - xV_{n-2}(x) - V_{n-3}(x)] + [V_{n+1}(x) - xV_{n-1}(x) - V_{n-2}(x)]i \\ &\quad + [V_{n+2}(x) - xV_n(x) - V_{n-1}(x)]\epsilon + [V_{n+3}(x) - xV_{n+1}(x) - V_n(x)]h = 0. \end{aligned}$$

As (V_n) is a Van Der Laan sequence. Thus we get

$$V_n^{[H]}(x) - xV_{n-2}^{[H]}(x) - V_{n-3}^{[H]}(x) = 0. \blacksquare$$

Theorem (3.12). The generating function for theVan Der Laan Hybrinomial sequence $\left\{V_n^{[H]}(x)\right\}$ is

$$\sum_{n=0}^{\infty} V_n^{[H]}(x)t^n = \frac{V_0^{[H]}(x) + V_1^{[H]}(x)t + \left(V_2^{[H]}(x) - xV_0^{[H]}(x)\right)t^2}{1 - t^2 - t^3}.$$

Proof. Suppose that the generating function of the Van Der Laan Hybrinomial sequence $\left\{V_n^{[H]}(x)\right\}$ has the form

$$f(t) = \sum_{n=0}^{\infty} V_n^{[H]}(x)t^n = V_0^{[H]}(x) + V_1^{[H]}(x)t + V_2^{[H]}(x)t^2 + V_3^{[H]}(x)t^3 + \cdots \quad .$$

Then we have

$$xt^2 f(t) = xV_0^{[H]}(x)t^2 + xV_1^{[H]}(x)t^3 + xV_2^{[H]}(x)t^4 + xV_3^{[H]}(x)t^5 + \cdots \quad ,$$

And

$$t^3 f(t) = V_0^{[H]}(x)t^3 + V_1^{[H]}(x)t^4 + V_2^{[H]}(x)t^5 + V_3^{[H]}(x)t^6 + \cdots \ .$$

Thus we obtain

$$\begin{aligned} &f(t) - t^2 f(t) - t^3 f(t) \\ &\quad = \left(V_0^{[H]}(x) + V_1^{[H]}(x)t + V_2^{[H]}(x)t^2 + V_3^{[H]}(x)t^3 + \cdots\right) - x\left(V_0^{[H]}(x)t^2\right. \\ &\quad \left. + V_1^{[H]}(x)t^3 + V_2^{[H]}(x)t^4 + V_3^{[H]}(x)t^5 + \cdots\right) - \left(V_0^{[H]}(x)t^3 + V_1^{[H]}(x)t^4\right. \\ &\quad \left. + V_2^{[H]}(x)t^5 + V_3^{[H]}(x)t^6 + \cdots\right) \\ &\quad = \left(V_0^{[H]}(x) + V_1^{[H]}(x)t\right) + \left(V_2^{[H]}(x) - xV_0^{[H]}(x)\right)t^2 \\ &\quad + \left(V_3^{[H]}(x) - xV_1^{[H]}(x) - V_0^{[H]}(x)\right)t^2 + \cdots \\ &\quad + \left(V_n^{[H]}(x) - xV_{n-2}^{[H]}(x) - V_{n-3}^{[H]}(x)\right)t^n + \cdots. \end{aligned}$$

By lemma (3.11) we have $V_n^{[H]}(x) - xV_{n-2}^{[H]}(x) - V_{n-3}^{[H]}(x) = 0$. So we obtain
$f(t) - t^2 f(t) - t^3 f(t) = V_0^{[H]}(x) + V_1^{[H]}(x)t + \left(V_2^{[H]}(x) - xV_0^{[H]}(x)\right)t^2.$
Thus we get
$f(t)(1 - t^2 - t^3) = i + h + (1 + \epsilon + h)t + (i + \epsilon + x - x(i + h))t^2.$
Consequently we have

$$\sum_{n=0}^{\infty} V_n^{[H]}(x)t^n = \frac{V_0^{[H]}(x) + V_1^{[H]}(x)t + \left(V_2^{[H]}(x) - xV_0^{[H]}(x)\right)t^2}{1 - t^2 - t^3}. \blacksquare$$

Corrolary (3.13). The exponential generating function for the Van Der Laan Hybrinomial sequence $\left\{V_n^{[H]}(x)\right\}$ is

$$\sum_{n=0}^{\infty} V_n^{[H]}(x)\frac{t^n}{n!}$$
$$\alpha\left(\frac{1 + \alpha i + \alpha^2\epsilon + \alpha^3 h}{(\alpha - \beta)(\alpha - \gamma)}\right)e^{\alpha t} + \beta\left(\frac{1 + \beta i + \beta^2\epsilon + \beta^3 h}{(\beta - \alpha)(\beta - \gamma)}\right)e^{\beta t} + \gamma\left(\frac{1 + \gamma i + \gamma^2\epsilon + \gamma^3 h}{(\gamma - \alpha)(\gamma - \beta)}\right)e^{\gamma t},$$

where α, β, γ are the roots of the equation $t^3 - xt - 1 = 0$.

Proof. By using the Binet like formula for the Van der Laan Hybrinomial sequence $\left\{V_n^{[H]}(x)\right\}$ we have $V_n^{[H]}(\mathrm{x}) =$

$$\left(\frac{1 + \alpha i + \alpha^2\epsilon + \alpha^3 h}{(\alpha - \beta)(\alpha - \gamma)}\right)\alpha^{n+1} + \left(\frac{1 + \beta i + \beta^2\epsilon + \beta^3 h}{(\beta - \alpha)(\beta - \gamma)}\right)\beta^{n+1} + \left(\frac{1 + \gamma i + \gamma^2\epsilon + \gamma^3 h}{(\gamma - \alpha)(\gamma - \beta)}\right)\gamma^{n+1}.$$

Thus we have

$$\sum_{n=0}^{\infty} V_n^{[H]}(x)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left[\left(\frac{1 + \alpha i + \alpha^2\epsilon + \alpha^3 h}{(\alpha - \beta)(\alpha - \gamma)}\right)\alpha^{n+1} + \left(\frac{1 + \beta i + \beta^2\epsilon + \beta^3 h}{(\beta - \alpha)(\beta - \gamma)}\right)\beta^{n+1} + \left(\frac{1 + \gamma i + \gamma^2\epsilon + \gamma^3 h}{(\gamma - \alpha)(\gamma - \beta)}\right)\gamma^{n+1}\right]\frac{t^n}{n!},$$

Therefore we can write

$$\sum_{n=0}^{\infty} V_n^{[H]}\frac{t^n}{n!} = \sum_{n=0}^{\infty}(\alpha k_1 {r_1}^n + \beta k_2 {r_2}^n + \gamma k_3 {r_3}^n)\frac{t^n}{n!}$$
$$= \alpha\left(\frac{1 + \alpha i + \alpha^2\epsilon + \alpha^3 h}{(\alpha - \beta)(\alpha - \gamma)}\right)\sum_{n=0}^{\infty}\frac{(\alpha t)^n}{n!} + \beta\left(\frac{1 + \beta i + \beta^2\epsilon + \beta^3 h}{(\beta - \alpha)(\beta - \gamma)}\right)\sum_{n=0}^{\infty}\frac{(\beta t)^n}{n!} + \gamma\left(\frac{1 + \gamma i + \gamma^2\epsilon + \gamma^3 h}{(\gamma - \alpha)(\gamma - \beta)}\right)\sum_{n=0}^{\infty}\frac{(\gamma t)^n}{n!}$$
$$= \alpha\left(\frac{1 + \alpha i + \alpha^2\epsilon + \alpha^3 h}{(\alpha - \beta)(\alpha - \gamma)}\right)e^{\alpha t} + \beta\left(\frac{1 + \beta i + \beta^2\epsilon + \beta^3 h}{(\beta - \alpha)(\beta - \gamma)}\right)e^{\beta t} + \gamma\left(\frac{1 + \gamma i + \gamma^2\epsilon + \gamma^3 h}{(\gamma - \alpha)(\gamma - \beta)}\right)e^{\gamma t}.$$

Thus the proof is completed.

4 CONCLUSION

In this paper we introduced the Van Der Laan Hybrinomial sequence. We presented the Binet-like formula, partial sum, generating function, exponential generating function, character and norm of this sequence. In addition we investigated some properties of this sequence.

Because of the application of particular number sequence in matrix algebras and combinatorial theory, the subject of this paper has the potential to motivate young researchers to introduced new number sequence related to this sequence. Since the complex, hyperbolic and dual number have several applications in the areas of mathematics and physics hence the subject of our paper is beneficial in these areas of sciences.

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