On Circulant Matrix involving Pell-Narayana sequence

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ABSTRACT

In this paper the Pell-Narayana sequence is introduced. Some properties and identities about this sequence are represented. Also the eigenvalues and determinant of circulant matrix involving Pell-Narayana sequence are computed.

KEYWORDS: Narayana sequence, Pell numbers, generating function, circulant matrix, determinant.

1 INTRODUCTION

Recently many authors have considered special number sequences like as the Narayana sequence because it has the close relation to the Fibonacci sequence. Also this sequence has many applications in Data coding and cryptography.

Narayana [1],[11] was an Indian mathematician who lived in the 14th century. He proposed the following problem: A cow produces one calf every year. Beginning in its fourth year, each calf produces one calf at the beginning of each year. How many cows are there totally after, 20?

By considering Narayana sequence according to the definition of Fibonacci's Rabbit problem, the Narayana[11] sequence (N_n) is defined by the recursion relation

$$N_{n+3} = N_{n+2} + N_n \text{for all } n \geq 3,$$

With initial values $N_0 = 2,\ N_1 = 3, N_2 = 4$.The first values of (N_n) are:

$$2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129,\ 189, 277.$$

Pell [9] sequence(P_n) is defined by the recursive relation

$$P_n = 2P_{n-1} + P_{n-2},$$

with the initial values$P_0 = 0, P_1 = 1$. The first values of (P_n) are:

0, 1 , 2 , 5 , 12 , 29 , 70 , 169 , 408 , 985 , 2378 , 5741.

In paper [10], the authors investigated the eigenvalues and determinant of special circulant matrix involving (k,h)-Jacobsthal sequence and (k,h)-Jacobsthal-like sequence. In this paper firstly the Pell-Narayana sequence is introduced. The Binet-like-formula, partial sum and generating function related to this sequence are represented. Some identities with some examples about this sequence are given. Finally the eigenvalues and determinant of circulant matrix involving Pell-Narayana sequence are represented.

For more information about Pell sequence, Pell Lucas sequence, Narayana sequence and some generalizations of these sequences we refer to[3-6], [8-10] and [12-13].

2 PELL – NARAYANA SEQUENCE

Definition (2.1). We define the Pell-Narayana sequence(PN_r) by the recursive relation

$$PN_r = 2PN_{r-1} + PN_{r-3} \quad (2.1.1)$$

With the initial values $PN_0 = 0, PN_1 = 1, PN_2 = 1.$ The first values of Pell-Narayana sequence are:

$0, 1, 1, 2, 5, 11, 24, 53, 117, 258, 569, 1255, 2768, 6105, 13465, 29698, 65501, 144967.$

Remark (2.2). Pell-Narayana sequence (PN_r) has characteristic equation $x^3 - 2x^2 - 1 = 0$. From the Cardano´s formula for the cubic equation $x^3 - 2x^2 - 1 = 0$ we can see that this equation has one real root α and two complex roots β, γ where

$$\alpha = \sqrt[3]{\frac{43}{54} + \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}} + \sqrt[3]{\frac{43}{54} - \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}} - \left(\frac{-2}{3}\right) \approx 2.205569904302,$$

$$\beta = \frac{-\left(\sqrt[3]{\frac{43}{54} + \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}} + \sqrt[3]{\frac{43}{54} - \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}}\right)}{2} + \left(\frac{2}{3}\right) + \frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{43}{54} + \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}} - \sqrt[3]{\frac{43}{54} - \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}}\right)i \approx -0.102784715 + (0.665469511)i,$$

And

$$\gamma = \frac{-\left(\sqrt[3]{\frac{43}{54} + \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}} + \sqrt[3]{\frac{43}{54} - \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}}\right)}{2} + \left(\frac{2}{3}\right) - \frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{43}{54} + \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}} - \sqrt[3]{\frac{43}{54} - \sqrt[2]{\left(\frac{-4}{9}\right)^3 + \left(\frac{43}{54}\right)^2}}\right)i \approx -0.102784715 - (0.665469511)i,$$

Where $i = \sqrt{-1}$.

Theorem (2.3). The generating function for the Pell-Narayana sequence (PN_r) is

$$\sum_{r=0}^{\infty} PN_r x^r = \frac{x - x^2}{1 - 2x - x^3}.$$

Proof. Suppose that the generating function for the Pell-Narayana sequence (PN_r) has the form

$$g(x) = \sum_{n=0}^{\infty} PN_r x^n = PN_0 + PN_1 x + PN_2 x^2 + PN_3 x^3 + \cdots + PN_r x^r + \cdots \quad .$$

Then we have

$$2xg(x) = 2xPN_0 + 2PN_1x^2 + 2PN_2x^3 + 2PN_3x^4 + \cdots + 2PN_rx^{r+1} + \cdots \ ,$$

And

$$x^3g(x) = PN_0x^3 + PN_1x^4 + PN_2x^5 + PN_3x^6 + \cdots + PN_rx^{r+3} + \cdots .$$

Thus we obtain

$g(x) - 2xg(x) - x^3g(x) =(PN_0 + PN_1x + PN_2x^2 + PN_3x^3 + \cdots) - (2xPN_0 + 2PN_1x^2 + 2PN_2x^3 + 2PN_3x^4 + \cdots \ 2PN_rx^{r+1} + \cdots \ ) - (PN_0x^3 + PN_1x^4 + PN_2x^5 + PN_3x^6 + \cdots + PN_rx^{r+3} + \cdots)$

$= PN_0 +(PN_1 - 2PN_0)x + (PN_2 - 2PN_1)x^2 + (PN_3 - 2PN_2 - PN_0)x^3 + \cdots + (PN_r - 2PN_{r-1} - PN_{r-3})x^n + \cdots.$

Therefore we get

$$g(x)(1 - 2x - x^3) = 0 + x + (1 - 2)x^2 + 0 = x - x^2.$$

Consequently

$$\sum_{r=0}^{\infty} PN_rx^r = \frac{x - x^2}{1 - 2x - x^3}.$$

Theorem (2.4). Let $r \geq 0$ be an integer. Then the Binet-like formula for the Pell-Narayana sequence(PN_r) is

$$PN_r = \frac{(\alpha - 1)}{(\alpha - \beta)(\alpha - \gamma)}\alpha^r + \frac{(\beta - 1)}{(\beta - \alpha)(\beta - \gamma)}\beta^r + \frac{(\gamma - 1)}{(\gamma - \alpha)(\gamma - \beta)}\gamma^r.$$

Where α, β, γ are the roots of the equation $x^3 - 2x^2 - 1 = 0$.

Proof. From remark (2.2) we see that the cubic equation $f(x) = x^3 - 2x^2 - 1 = 0$ has three distinct roots α, β, γ. Hence $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are the roots $h(x) = f\left(\frac{1}{x}\right) = 1 - 2x - x^3$. In exact we have

$$h(x) = 1 - 2x - x^3 = (1 - \alpha x)(1 - \beta x)(1 - \gamma x).$$

According to the generating function of Pell-Narayana sequence we have

$$g(x) = \frac{x - x^2}{1 - 2x - x^3} = \frac{A}{1 - \alpha x} + \frac{B}{1 - \beta x} + \frac{C}{1 - \gamma x}$$

$$= A\sum_{r=0}^{\infty}(\alpha x)^r + B\sum_{r=0}^{\infty}(\beta x)^r + C\sum_{r=0}^{\infty}(\gamma x)^r . \qquad (3.4.1)$$

Thus we have

$$g(x) = \frac{x - x^2}{1 - 2x - x^3} = \frac{A(1 - \beta x)(1 - \gamma x) + B(1 - \alpha x)(1 - \gamma x) + C(1 - \alpha x)(1 - \beta x)}{(1 - \alpha x)(1 - \beta x)(1 - \gamma x)}.$$

Therefore by comparison of the left and right sides of this equality we get that

$$x - x^2 = A(1 - \beta x)(1 - \gamma x) + B(1 - \alpha x)(1 - \gamma x) + C(1 - \alpha x)(1 - \beta x).$$

If we set $x = \frac{1}{\alpha}$ we find that

$$\frac{1}{\alpha} - \frac{1}{\alpha^2} = A\left(1 - \frac{\beta}{\alpha}\right)\left(1 - \frac{\gamma}{\alpha}\right).$$

Consequently we get

$$A = \frac{(\alpha - 1)}{(\alpha - \beta)(\alpha - \gamma)}.$$

Similarly we get

$$B = \frac{(\beta - 1)}{(\beta - \alpha)(\beta - \gamma)} \quad , C = \frac{(\gamma - 1)}{(\gamma - \alpha)(\gamma - \beta)} .$$

By (3.4.1) we obtain that

$$g(x) = \sum_{r=0}^{\infty}\frac{(\alpha-1)\alpha^r}{(\alpha-\beta)(\alpha-\gamma)}x^r + \sum_{r=0}^{\infty}\frac{(\beta-1)\beta^r}{(\beta-\alpha)(\beta-\gamma)}x^r + \sum_{r=0}^{\infty}\frac{(\gamma-1)\gamma^r}{(\gamma-\alpha)(\gamma-\beta)}x^r$$

$$= \sum_{r=0}^{\infty}\left[\frac{(\alpha-1)\alpha^r}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(\beta-1)\beta^r}{(\beta-\alpha)(\beta-\gamma)} + \frac{(\gamma-1)\gamma^r}{(\gamma-\alpha)(\gamma-\beta)}\right]x^r.$$

Consequently we obtain

$$PN_r = \frac{(\alpha-1)}{(\alpha-\beta)(\alpha-\gamma)}\alpha^r + \frac{(\beta-1)}{(\beta-\alpha)(\beta-\gamma)}\beta^r + \frac{(\gamma-1)}{(\gamma-\alpha)(\gamma-\beta)}\gamma^r.$$

Theorem (2.5). Let $r \geq 0$ be an integer and k be an arbitrary integer. Then

(a) $PN_{r+k} + PN_{r-k} = \frac{(\alpha-1)(\alpha^{2k}+1)}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{r-k} + \frac{(\beta-1)(\beta^{2k}+1)}{(\beta-\alpha)(\beta-\gamma)}\beta^{r-k} + \frac{(\gamma-1)(\gamma^{2k}+1)}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{r-k}$,

(b) $PN_{r+k} - PN_{r-k} = \frac{(\alpha-1)(\alpha^{2k}-1)}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{r-k} + \frac{(\beta-1)(\beta^{2k}-1)}{(\beta-\alpha)(\beta-\gamma)}\beta^{r-k} + \frac{(\gamma-1)(\gamma^{2k}-1)}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{r-k}$.

Proof. They can be proved by direct calculations from theorem (3.4).

Corollary (2.6). From theorem (2.5) for $k = 1$ we have

(a) $PN_{r+1} + PN_{r-1} = \frac{(\alpha-1)(\alpha^2+1)}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{r-1} + \frac{(\beta-1)(\beta^2+1)}{(\beta-\alpha)(\beta-\gamma)}\beta^{r-1} + \frac{(\gamma-1)(\gamma^2+1)}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{r-1}$,

(b) $PN_{r+1} - PN_{r-1} = \frac{(\alpha-1)^2(\alpha+1)}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{r-1} + \frac{(\beta-1)^2(\beta+1)}{(\beta-\alpha)(\beta-\gamma)}\beta^{r-1} + \frac{(\gamma-1)^2(\gamma+1)}{(\gamma-\alpha)(\gamma-\beta)}\gamma^{r-1}$.

Lemma (2.7). Let $r \geq 0$ be an integer. Then

$$\sum_{r=0}^{n} PN_r = \frac{1}{2}[PN_{r+3} - PN_{r+1} - PN_{r+2} + 1].$$

Proof. It can be proved according to the definition of Pell-Narayana sequence we now

3 MOR IDENTITIES ABOUT PELL NARAYANA SEQUENCE

Theorem (3.1). Let $n \geq 0$ be an integer and $\varphi = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}$. Then

$$\begin{bmatrix} PN_n \\ PN_{n+1} \\ PN_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \varphi^n \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Proof. We prove this theorem by mathematical induction on n.
For $n = 1$we have

$$\varphi^1\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\1&0&2\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}1\\1\\2\end{bmatrix}=\begin{bmatrix}PN_1\\PN_2\\PN_3\end{bmatrix}.$$

Thus the result is true for $n=1$. Now suppose that the result is true for $n=k$. Hence we have

$$\begin{bmatrix}PN_k\\PN_{k+1}\\PN_{k+2}\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\1&0&2\end{bmatrix}^k\begin{bmatrix}0\\1\\1\end{bmatrix}=\varphi^k\begin{bmatrix}0\\1\\1\end{bmatrix}.$$

Then we have

$$\varphi^{k+1}\begin{bmatrix}0\\1\\1\end{bmatrix}=\varphi\varphi^k\begin{bmatrix}0\\1\\1\end{bmatrix}=\varphi\begin{bmatrix}PN_k\\PN_{k+1}\\PN_{k+2}\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\1&0&2\end{bmatrix}\begin{bmatrix}PN_k\\PN_{k+1}\\PN_{k+2}\end{bmatrix}=\begin{bmatrix}PN_{k+1}\\PN_{k+2}\\PN_{k+3}\end{bmatrix}.$$

Therefore the result is true for $n=k$. Consequently by induction the result is true for every n. This proves the theorem.

Remark (3.2). As we know the characteristic polynomial of the recursive relation $PN_r=2PN_{r-1}+PN_{r-3}$ is $p(x)=x^3-2x^2-1=0$. This polynomial can be written as

$$p(x)=\det(xI-\varphi)=0, \text{ where } I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \text{ and } \varphi=\begin{bmatrix}0&1&0\\0&0&1\\1&0&2\end{bmatrix}.$$

From the well-known Cayley Hamilton theorem in matrix algebra we have $p(\varphi)=0$. Thus we have

$$\varphi^3-2\varphi^2-I=0. \quad (3.2.1)$$

Theorem (3.3). Let $=\begin{bmatrix}0&1&0\\0&0&1\\1&0&2\end{bmatrix}$, then

$$I=\varphi^3-2\varphi^2=\varphi^3-2\varphi^5+4\varphi^4,$$

And

$$\varphi^n=\varphi^{n+3}-2\varphi^{n+5}+4\varphi^{n+4}.$$

Proof. According to the remark (3.2) we have

$$\begin{aligned}I&=\varphi^3-2\varphi^2=\varphi^2(\varphi-2I)=\varphi^2\big(\varphi-2(\varphi^3-2\varphi^2)\big)=\varphi^2(\varphi-2\varphi^3+4\varphi^2)\\&=\varphi^3-2\varphi^5+4\varphi^4.\end{aligned}$$

Thus

$$I=\varphi^3-2\varphi^5+4\varphi^4.$$

This proves the first equality. Multiplying both sides of the above equality by φ^n, we obtain

$$\varphi^n=\varphi^{n+3}-2\varphi^{n+5}+4\varphi^{n+4}. \quad (3.3.1)$$

Thus the proof is completed.

Corollary (3.4). Let $r\geq 0$ be an integer. Then

$$\varphi^{n+5}=\frac{1}{2}[4\varphi^{n+4}+\varphi^{n+3}-\varphi^n]. \quad (3.4.1)$$

According to this corollary we have the following interesting example and Theorem about the Pell-Narayana sequence(PN_r).

Example (3.5). From the first values of Pell-Narayana sequence (PN_r) we have

$$24 = \frac{1}{2}[4 \times 11 + 5 - 1].$$

In exact we have

$$PN_{1+5} = \frac{1}{2}[4PN_5 + PN_4 - PN_1] = \frac{1}{2}[4PN_{1+4} + PN_{1+3} - PN_1].$$

Therefore by induction we have the following identity about the Pell-Narayana sequence(PN_r).

Theorem (3.6). Let $r \geq 0$ be an integer. Then

$$PN_{r+5} = \frac{1}{2}[4PN_{r+4} + PN_{r+3} - PN_r].$$

Proof. We prove this theorem by mathematical induction on n. According to the last example we see that $PN_{1+5} = \frac{1}{2}[4PN_{1+4} + PN_{1+3} - PN_1]$. Then if we assume that $PN_{t+5} = \frac{1}{2}[4PN_{t+4} + PN_{t+3} - PN_r]$ for all $t < n$. Then we have

$$\begin{aligned}
PN_{n+5} &= 2PN_{n+4} + PN_{n+2} \\
&= 2 \times \frac{1}{2}[4PN_{n+3} + PN_{n+2} - PN_{n-1}] + \frac{1}{2}[4PN_{n+1} + PN_n - PN_{n-3}] \\
&= 4PN_{n+3} + PN_{n+2} - PN_{n-1} + 2PN_{n+1} + \frac{1}{2}PN_n - \frac{1}{2}PN_{n-3} \\
&= \frac{1}{2}(4[2PN_{n+3} + PN_{n+1}]) + \frac{1}{2}(2PN_{n+2} + PN_n) - \frac{1}{2}(2PN_{n-1} + PN_{n-3}) \\
&= \frac{1}{2}[4PN_{n+4} + PN_{n+3} - PN_n].
\end{aligned}$$

Thus the result is true for all n.

Theorem (3.7). Let $r, n \geq 0$ be integer. Then

$$\varphi^{n+r} = \varphi^{n+r+6} + 8\varphi^{n+r+7} + 12\varphi^{n+r+8} - 16\varphi^{n+r+9} + 4\varphi^{n+r+10}.$$

Proof. By theorem (3.4) we have

$$\varphi^n = \varphi^{n+3} - 2\varphi^{n+5} + 4\varphi^{n+4}.$$

Hence

$$\begin{aligned}
\varphi^{n+r} = \varphi^n\varphi^r &= (\varphi^{n+3} - 2\varphi^{n+5} + 4\varphi^{n+4})(\varphi^{r+3} - 2\varphi^{r+5} + 4\varphi^{r+4}) \\
&= \varphi^{n+r+6} - 2\varphi^{n+r+8} + 4\varphi^{n+r+7} - 2\varphi^{n+r+8} + 4\varphi^{n+r+10} - 8\varphi^{n+r+9} \\
&+ 4\varphi^{n+r+7} - 8\varphi^{n+r+9} + 16\varphi^{n+r+8} \\
&= \varphi^{n+r+6} + 8\varphi^{n+r+7} + 12\varphi^{n+r+8} - 16\varphi^{n+r+9} + 4\varphi^{n+r+10}.
\end{aligned}$$

Thus the proof is completed.

Corollary (3.8). Let $n \geq 0$ be an integer. Then

(a) $\varphi^{2n} = \varphi^{2n+6} + 8\varphi^{2n+7} + 12\varphi^{2n+8} - 16\varphi^{2n+9} + 4\varphi^{2n+10}$,
(b) $\varphi^{2n+9} = \frac{1}{16}(\varphi^{2n+6} + 8\varphi^{2n+7} + 12\varphi^{2n+8} + 4\varphi^{2n+10} - \varphi^{2n})$.

Proof. They can be derived by taking $r = n$ from theorem (3.7).

4 CIRCULANT MATRIX INVOLVING PELL-NARAYANA SEQUENCE

Definition (4.1). A matrix $C = [c_{i,j}] \in M_{n\times n}$is called a Circulant matrix if it is of the form

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \end{bmatrix}$$

A circulant matrix $C = \left[c_{i,j}\right]$ can be written by $C = Circ(c_0, c_1, \cdots, c_{n-1})$.

Lemma (4.2). [6] Let $C = Circ(c_0, c_1, \cdots, c_{n-1})$ be a $n \times n$ circulant matrix. Then we have

$$\rho_j(C) = \sum_{k=0}^{n-1} c_k w^{-jk},$$

Where ρ_j for $j = 0,1,2, \cdots, n-1$ is the eigenvalue of the circulant matrix C and $w = e^{\frac{2\pi i}{n}}$ $, i = \sqrt{-1}$.

Remark (4.3). Since α, β, γ are the roots of the cubic equation $x^3 - 2x^2 - 1 = 0$. By direct calculation we can prove that

(a) $\alpha + \beta + \gamma = 2$,

(b) $\alpha\beta\gamma = 1$,

(c) $\alpha\beta + \alpha\gamma + \beta\gamma = 0$.

Lemma (4.4). Suppose that α, β, γ are the roots of the equation $x^3 - 2x^2 - 1 = 0$ and

$$k_1 = \frac{\alpha - 1}{(\alpha - \beta)(\alpha - \gamma)}\ , k_2 = \frac{\beta - 1}{(\beta - \alpha)(\beta - \gamma)}\ , k_3 = \frac{\gamma - 1}{(\gamma - \alpha)(\gamma - \beta)}\ ,$$

Then we have

(a) $k_1 + k_2 + k_3 = 0$,

(b) $k_1 + k_2 = \frac{1-\gamma}{(\alpha-\gamma)(\beta-\gamma)}$, $k_1 + k_3 = \frac{1-\beta}{(\alpha-\beta)(\gamma-\beta)}$, $k_2 + k_3 = \frac{1-\alpha}{(\beta-\alpha)(\gamma-\alpha)}$,

(c) $(k_1 + k_3)\beta + (k_2 + k_3)\alpha + (k_1 + k_2)\gamma = -1$,

(d) $\frac{k_1}{\alpha} + \frac{k_2}{\beta} + \frac{k_3}{\gamma} = -1$.

Proof. They can be proved directly by some computations according to the definition of k_1, k_2, k_3 and properties of α, β, γ.

Theorem (4.5). Let $C = Cir(PN_0, PN_1, \cdots, PN_{n-1})$ be a $n \times n$ circulant matrix whose entries are the Pell-Narayana sequence(PN_n). Then the eigenvalues of C are

$$\rho_j(\mathrm{C}) = \frac{(PN_{n-1} + 1)w^{-2j} + (PN_{n-2} - 1)w^{-j} + PN_n}{w^{-3j} + w^{-2j} - 1}, (\text{ for } j = 0,1,2, \cdots, n-1)$$

where $i = \sqrt{-1}$, and $w = e^{\frac{2\pi i}{n}}$.

Proof. By lemma (4.2) for the eigenvalues of circulant matrix $C = Cir(PN_0, PN_1, \cdots, PN_{n-1})$ we have

$$\rho_j(C) = \sum_{k=0}^{n-1} PN_k w^{-jk}$$

$$= \sum_{k=0}^{n-1} \left[\frac{(\alpha-1)}{(\alpha-\beta)(\alpha-\gamma)} \alpha^k + \frac{(\beta-1)}{(\beta-\alpha)(\beta-\gamma)} \beta^k + \frac{(\gamma-1)}{(\gamma-\alpha)(\gamma-\beta)} \gamma^k \right] w^{-jk}$$

$$= \frac{(\alpha-1)}{(\alpha-\beta)(\alpha-\gamma)} \sum_{k=0}^{n-1} \alpha^k w^{-jk} + \frac{(\beta-1)}{(\beta-\alpha)(\beta-\gamma)} \sum_{k=0}^{n-1} \beta^k w^{-jk}$$

$$+ \frac{(\gamma-1)}{(\gamma-\alpha)(\gamma-\beta)} \sum_{k=0}^{n-1} \gamma^k w^{-jk}.$$

Therefore by taking $k_1 = \frac{\alpha-1}{(\alpha-\beta)(\alpha-\gamma)}$, $k_2 = \frac{\beta-1}{(\beta-\alpha)(\beta-\gamma)}$, $k_3 = \frac{\gamma-1}{(\gamma-\alpha)(\gamma-\beta)}$ we have

$$\rho_j(\mathrm{C}) = k_1 \left(\frac{(\alpha w^{-j})^n - 1}{\alpha w^{-j} - 1} \right) + k_2 \left(\frac{(\beta w^{-j})^n - 1}{\beta w^{-j} - 1} \right) + k_3 \left(\frac{(\gamma w^{-j})^n - 1}{\gamma w^{-j} - 1} \right)$$

$$= k_1 \left(\frac{\alpha^n - 1}{\alpha w^{-j} - 1} \right) + k_2 \left(\frac{\beta^n - 1}{\beta w^{-j} - 1} \right) + k_3 \left(\frac{\gamma^n - 1}{\gamma w^{-j} - 1} \right)$$

$$= \frac{k_1(\alpha^n - 1)(\beta w^{-j} - 1)(\gamma w^{-j} - 1) + k_2(\beta^n - 1)(\alpha w^{-j} - 1)(\gamma w^{-j} - 1)}{(\alpha w^{-j} - 1)(\beta w^{-j} - 1)(\gamma w^{-j} - 1)}$$

$$+ \frac{k_3(\gamma^n - 1)(\alpha w^{-j} - 1)(\beta w^{-j} - 1)}{(\alpha w^{-j} - 1)(\beta w^{-j} - 1)(\gamma w^{-j} - 1)}.$$

Thus we get

$$\rho_j(\mathrm{C}) = \frac{-(k_1 + k_2 + k_3) + (k_1\alpha^n + k_2\beta^n + k_3\gamma^n) + (k_1\alpha^n\beta\gamma + k_2\beta^n\alpha\,\gamma + k_3\gamma^n\alpha\beta)w^{-2j}}{(\alpha\beta\gamma)w^{-3j} - (\alpha\beta + \alpha\gamma + \beta\gamma)w^{-2j} + (\alpha + \beta + \gamma)w^{-j} - 1}$$

$$+ \frac{-(k_1\alpha^n\beta + k_2\beta^n\alpha + k_3\gamma^n\alpha)w^{-j} - (k_1\alpha^n\gamma + k_2\beta^n\gamma + k_3\gamma^n\beta)w^{-j}}{(\alpha\beta\gamma)w^{-3j} - (\alpha\beta + \alpha\gamma + \beta\gamma)w^{-2j} + (\alpha + \beta + \gamma)w^{-j} - 1}$$

$$+ \frac{-(k_1\beta\gamma + k_2\alpha\gamma + k_3\alpha\beta)w^{-2j} + (k_1\beta + k_2\alpha + k_3\alpha + k_1\gamma + k_2\gamma + k_3\beta)w^{-j}}{(\alpha\beta\gamma)w^{-3j} - (\alpha\beta + \alpha\gamma + \beta\gamma)w^{-2j} + (\alpha + \beta + \gamma)w^{-j} - 1}.$$

According to the remark (4.3) and lemma (4.4), after some computations we get

$$\rho_j(\mathrm{C}) = \frac{(PN_{n-1} + 1)w^{-2j} + (PN_{n+1} - 2PN_n - 1)w^{-j} + PN_n}{w^{-3j} + w^{-2j} - 1}$$

$$= \frac{(PN_{n-1} + 1)w^{-2j} + (PN_{n-2} - 1)w^{-j} + PN_n}{w^{-3j} + w^{-2j} - 1}.$$

Thus the proof is completed.

Example (4.6). The following table represents the eigenvalues of $C = Cir(PN_0, PN_1, \cdots, PN_{n-1})$ for some values of n.

n	**Eigenvalues of** $C = Cir(PN_0, PN_1, \cdots, PN_{n-1})$
2	-1 1
3	4 -2 $-1+i$, $-1-i$
4	9 $0.572949 - 4.39201i$ $0.572949 + 4.39201i$ $-3.92705 + 1.40008i$ $-3.92705 - 1.40008i$
5	20 $1 + 12.1244i$ $1 - 12.1244i$ $-7 + 5.19615i$ $-7 - 5.19615i$ -8
6	44 $6.61021 + 29.0331i$ $6.61021 - 29.0331i$ $-12.0102 + 15.739i$ $-12.0102 - 15.739i$ $16.6 + 5.08579i$ $16.6 - 5.08579i$
7	97 $23.9914 + 66.1335i$ $23.9914 - 66.1335i$ $-20 + 43i$ $-20 - 43i$ $-33.9914 + 20.1335i$ $-33.9914 - 20.1335i$ -37

Lemma (4.7). Let $\mathrm{x}, \mathrm{y}, \mathrm{z}$ be real numbers and $n > 0$ be an integer. Then

$$\prod_{k=0}^{n-1}(x - yw^{-k} + zw^{-2k}) = x^n\left(1 - \left(\frac{y - \sqrt{y^2 - 4xz}}{2x}\right)^n - \left(\frac{y + \sqrt{y^2 - 4xz}}{2x}\right)^n + \left(\frac{z}{x}\right)^n\right)$$

$$= x^n + z^n - \left[\left(\frac{y - \sqrt{y^2 - 4xz}}{2}\right)^n + \left(\frac{y + \sqrt{y^2 - 4xz}}{2}\right)^n\right],$$

Where $w = e^{\frac{2\pi i}{n}}$.

Proof. See [2].

Lemma (4.8). Let $n > 0$ be an integer. Then

$$\prod_{j=0}^{n-1}\left(w^{-3j} + w^{-2j} - 1\right) = (-1)^n(-Q_{-n} - Q_n).$$

Where (Q_n) is the Perrin sequence that is defined by the recursive relation $Q_{n+3} = Q_{n+1} + Q_n$ with initial values $Q_0 = 3, Q_1 = 0, Q_2 = 2$ and sequence (Q_{-n}) is defined by recursive relation $Q_{-n} = Q_{-(n-1)} + Q_{-(n-3)}$.

Proof. See [2].

Theorem (4.9). Let $C = Cir(PN_0, PN_1, \cdots, PN_{n-1})$ be a $n \times n$ circulant matrix whose entries are the Pell-Narayana sequence(PN_n). Then determinant of C is

$$\begin{aligned}\det(C) = \Bigg(& {PN_n}^r + (PN_{n-1} + 1)^r \\ & - \left[\left(\frac{(PN_{n-2} - 1) - \sqrt{(PN_{n-2} - 1)^2 - 4PN_n(PN_{n-1} + 1)}}{2}\right)^r \right. \\ & \left. + \left(\frac{(PN_{n-2} - 1) + \sqrt{(PN_{n-2} - 1)^2 - 4PN_n(PN_{n-1} + 1)}}{2}\right)^r\right]\Bigg) \\ & \times \left(\frac{1}{(-1)^n(Q_{-n} - Q_n)}\right).\end{aligned}$$

Proof. Let $\rho_0, \rho_1, \cdots, \rho_{r-1}$ are the eigenvalues of circulant matrix C. From a basic theorem in matrix algebra about the determinant of a matrix we have

$$\det(C) = \prod_{j=0}^{n-1} \rho_j$$

Therefore by theorem (4.5) we get

$$\begin{aligned}\det(C) &= \prod_{j=0}^{n-1} \rho_j \\ &= \prod_{j=0}^{n-1} \frac{(PN_{n-1} + 1)w^{-2j} + (PN_{n-2} - 1)w^{-j} + PN_n}{w^{-3j} + w^{-2j} - 1} \\ &= \prod_{j=0}^{n-1}\left[(PN_{n-1} + 1)w^{-2j} + (PN_{n-2} - 1)w^{-j} + PN_n\right] \\ &\times \left(\frac{1}{\prod_{j=0}^{r-1}(w^{-3j} + w^{-2j} - 1)}\right).\end{aligned}$$

Therefore by lemma (4.7) and lemma (4.8) we have

$$\det(C) = \Bigg(PN_n{}^r + (PN_{n-1}+1)^r - \left[\left(\frac{(PN_{n-2}-1) - \sqrt{(PN_{n-2}-1)^2 - 4PN_n(PN_{n-1}+1)}}{2} \right)^r + \left(\frac{(PN_{n-2}-1) + \sqrt{(PN_{n-2}-1)^2 - 4PN_n(PN_{n-1}+1)}}{2} \right)^r \right] \Bigg) \times \left(\frac{1}{(-1)^n (Q_{-n} - Q_n)} \right).$$

Example (4.10). The following table shows the determinant of $C = Cir(PN_0, PN_1, \cdots, PN_{n-1})$ for some values of n.

n	**Determinant of** $C = Cir(PN_0, PN_1, \cdots, PN_{n-1})$
2	-2
3	2
4	-16
5	3069
6	-1799680
7	4609034012
8	-62350489778837

5 CONCLUSION

In this paper we introduced the Pell-Narayana sequence. We obtained Binet-like formula of this sequence. We studied the generating function and partial sum of this sequence. We investigated some interesting identities and examples about this sequence. Also eigenvalues and determinant of circulant matrix involving this sequence are represented.

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