Numerical Solution for Solving of Two-Dimensional Linear Fredholm Integral Equations Using Boubaker Polynomial Bases

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ABSTRACT

In this paper, we have introduced a new collocation method based on Boubaker polynomials for approximate solution to the two-dimensional linear Fredholm integral equations of the second kind. The basic integral matrix is used by collocation points to reduce the integral equations to some linear algebraic system. Error analysis has been studied, and the validity and accuracy of the presented method are demonstrated through illustrative examples. Using the MATLAB software, the values of the examples in the tables and figures are given for comparing with modification of hat functions and Taylor matrix methods.

KEYWORDS: two-dimensional Fredholm integral equations, the basic matrix, terminate Boubaker polynomials series, collocation method.

1 INTRODUCTION

Analytical solution of two-dimensional integral equations is usually difficult and very complex. Therefore, numerical methods are needed to arrive at an acceptable solution. In this paper, we present a terminate collocation method for numerical solution of two-dimensional linear Fredholm integral equations in the following form

$$u(x,y) = g(x,y) + \int_a^b \int_c^d K(x,y,t,s)u(t,s)dsdt \quad , \quad (x,y)\epsilon D \tag{1}$$

Where $u(x,y)$ is unknown function, $g(x,y)$ and $K(x,y,t,s)$ functions, reapectively are continuous and defined in the intervals $D=[a,b]\times[c,d]$ and $E = D\times D$. The purpose of this paper is to find a solution for the Eq.(1) according to $(2D-TBPs)$method, so that for all (x,y) in D is defined by: $u(x,y)=\sum_{n=0}^{N}\sum_{m=0}^{M}\beta_{nm}\,\psi_m^n(x,y)$,where $\psi_m^n(x,y)(m,n=0,1,\dots)$ are two-dimensional Boubaker polynomials, that are defined as follows: $\psi_m^n(x,y)=B_n(x)B_m(y)$. β_{mn} is to two dimensional Boubaker unknowns confficients, also n and m are polynomial of degrees also N,M are positive integres $(m\le M, n\le N)$. To get the numerical solution Eq.(1), we use the following collocation points

$$\begin{cases} x_i = a + \frac{b-a}{N} \cdot i \quad ; \quad i = 0,1, \ldots, N \\ y_j = c + \frac{d-c}{M} \cdot j \quad ; \quad j = 0,1, \ldots, M \end{cases} \tag{2}$$

the following successive relation holds for one-dimensional Boubaker polynomials [1]

$$B_k(x) = x \cdot B_{k-1}(x) - B_{k-2}(x) \qquad , \quad k > 2$$

Where$B_0(x) = 1, B_1(x) = x, B_2(x) = x^2 + 2$. A sequence of these integer non-orthogonal polynomials made by Boubaker as [3]:

$$B_n(x) = \sum_{k=0}^{\xi(n)} \left\{ \frac{(n-4k)}{n-k} C_{n-k}^{k} \right\} (-1)^k x^{n-2k}$$

Where $\xi(n) = \left\lfloor \frac{n}{2} \right\rfloor = \frac{2n+((-1)^n-1)}{4}$, and C_{n-k}^{k} is the binomial cofficients $\binom{n-k}{n}$

2 SOME PRELIMINARIES OF BOUBAKER POLYNOMIALS

We can convert Boubaker polynomials series into the matrix from

$$u(x,y) = \Psi(x,y) \cdot \beta \tag{3}$$

Where Ψ , β are defined as:

$$\Psi = [\psi_0^0, \ldots, \psi_M^0, \ldots, \psi_0^N, \ldots, \psi_M^N]$$
$$\beta = [\beta_{00}, \ldots, \beta_{0M}, \ldots, \beta_{N0}, \ldots, \beta_{NM}]^T$$

we have

$$\Psi(x,y) = X(x,y) \cdot Z^T \tag{4}$$

The row vector X is defined as :$X = [1, y, \ldots, y^M, x, xy, \ldots, xy^M, \ldots, x^N, \ldots, x^N y^M]$
Let $N = M$; if N is even, the Z square matrix are defined as:

$$Z = \begin{bmatrix} \gamma_{0,0} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \gamma_{1,0} & 0 & 0 & \cdots & 0 & 0 \\ \gamma_{2,1} & 0 & \gamma_{2,0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & \gamma_{(N-1)^2, \frac{(N-2)^2}{4}} & 0 & \gamma_{(N-1)^2, \frac{(N-4)^2}{4}} & \cdots & \gamma_{(N-1)^2, 0} & 0 \\ \gamma_{N^2, \frac{N^2}{4}} & 0 & \gamma_{N^2, \frac{(N-2)^2}{4}} & 0 & \cdots & 0 & \gamma_{N^2, 0} \end{bmatrix}$$

When $n = m$ and $p = q$, we show the elements of the matrix Z as $\gamma_{n,p}^{m,q} = \gamma_{n,p}$, so that

$$\psi_m^n(x,y)=\sum_{p=0}^{\xi(\mathrm{n})}\sum_{\mathrm{q}=0}^{\xi(m)}\gamma_{\mathrm{n,p}}^{\mathrm{m,q}}\mathrm{x}^{\mathrm{n}-2\mathrm{p}}\cdot\mathrm{y}^{\mathrm{m}-2\mathrm{q}}\quad;\quad\gamma_{\mathrm{n,p}}^{\mathrm{m,q}}=(-1)^{\mathrm{p+q}}\left\{\frac{(\mathrm{n}-4\mathrm{p})(\mathrm{m}-4\mathrm{q})}{(\mathrm{n}-\mathrm{p})(\mathrm{m}-\mathrm{q})}\mathrm{C}_{\mathrm{n-p}}^{\mathrm{p}}\mathrm{C}_{\mathrm{m-q}}^{\mathrm{q}}\right\}$$

$$\begin{bmatrix}\mathrm{q}=0,1,\dots,\left\lfloor\frac{m}{2}\right\rfloor & ; & \mathrm{m}=0,1,\dots,\mathrm{M}\\ \mathrm{p}=0,1,\dots,\left\lfloor\frac{n}{2}\right\rfloor & ; & \mathrm{n}=0,1,\dots,\mathrm{N}\end{bmatrix}$$

So Eq.(1) can be rewritten as

$$\mathrm{u}(\mathrm{x,y})=\mathrm{g}(\mathrm{x,y})+\mathrm{I}(\mathrm{x,y}) \tag{5}$$

Where

$$\mathrm{I}(\mathrm{x,y})=\int_{\mathrm{a}}^{\mathrm{b}}\int_{\mathrm{c}}^{\mathrm{d}}\mathrm{K}(\mathrm{x,y,,t,s})\mathrm{u}(\mathrm{t,s})\mathrm{dsdt} \tag{6}$$

Substituting collocation points Eq. (2) in Eq.(6), we can get

$$\mathrm{x}(\mathrm{x_i,y_j})=\mathrm{g}(\mathrm{x_i,y_j})+\mathrm{I}(\mathrm{x_i,y_j})\quad;\quad\mathrm{i}=0,1,\dots,\mathrm{N}\quad,\quad\mathrm{j}=0,1,\dots,\mathrm{M} \tag{7}$$

The above systems (7), can be written as , $\mathrm{u}=\mathrm{G}+\mathrm{I}$. Now, we want to find the matrix I.

We have $\mathrm{K}(\mathrm{x,y,t,s})=\sum_{=0}^{\mathrm{N}}\sum_{\mathrm{l}=0}^{\mathrm{M}}\mathrm{h_{k,l}}(\mathrm{x,y})\,\psi_{\mathrm{l}}^{\mathrm{k}}(\mathrm{t,s})$

therefore

$$\mathrm{K}(\mathrm{x,y,t,s})=\mathrm{H}(\mathrm{x,y})\cdot\Psi^T(t,s),\mathrm{H}(\mathrm{x,y})=\left[\mathrm{h_{00}}(\mathrm{x,y}),\dots,\mathrm{h_{0,M}}(\mathrm{x,y}),\dots,\mathrm{h_{N,0}}(\mathrm{x,y}),\dots,\mathrm{h_{N,M}}(\mathrm{x,y})\right] \tag{8}$$

Substituting the matrix forms of Eqs. (8), (3) in Eg. (6), the following matrix from is obtained

$$\mathrm{I}(\mathrm{x,y})=\int_a^b\int_c^d\mathrm{H}(x,y)\cdot\Psi^T(t,s)\cdot\Psi(t,s)\cdot\beta dsdt=\mathrm{H}(x,y)\cdot\left(\int_a^b\int_c^d\Psi^T(t,s)\cdot\Psi(t,s)dsdt\right)\cdot\beta$$

Put

$$\mathrm{Q}=\int_a^b\int_c^d\Psi^T(t,s)\cdot\Psi(t,s)\cdot\mathrm{dsdt} \tag{9}$$

So in summary

$$\mathrm{I}(\mathrm{x,y})=\mathrm{H}(\mathrm{x,y})\cdot\mathrm{Q}\cdot\beta \tag{10}$$

To calculate the elements of the Q matrix, put Eg.(4) in Eg.(9),si we have

$$\mathrm{Q}=\mathrm{Z}\cdot\int_{\mathrm{a}}^{\mathrm{b}}\int_{\mathrm{c}}^{\mathrm{d}}\mathrm{X}^{\mathrm{T}}(\mathrm{t,s})\cdot\mathrm{X}(\mathrm{t,s})\mathrm{dsdt}\cdot\mathrm{Z}^{\mathrm{T}} \tag{11}$$

According to Eq.(11), we can write

$$\mathrm{R}=\int_{\mathrm{a}}^{\mathrm{b}}\int_{\mathrm{c}}^{\mathrm{d}}\mathrm{X}^{\mathrm{T}}(\mathrm{t,s}).\mathrm{X}(\mathrm{t,s})\mathrm{dsdt}=[\mathrm{r_{ij}}] \tag{12}$$

Where $r_{ij} = \frac{(b^{i+j+1} - a^{i+j+1})(d^{i+j+1} - c^{i+j+1})}{(i+j+1)^2}$. Also, substituting Eq.(12) in Eq.(11), we have

$$Q = Z.R.Z^T \tag{13}$$

Ultimately, substituting Eq.(13) in Eq.(10), we get

$$I(x,y) = H(x,y).Z.R.Z^T.\beta \tag{14}$$

3 NUMERICAL IMPLEMENTATION

Now, substituting Eqs.(3), (5),(11) and (14) in Eq.(6), then by puting (2), we have

$$\{X(x_i, y_j).Z^T - H(x_i, y_j).Q\}.\beta = g(x_i, y_j) \tag{15}$$

Briefly, $\{X.Z^T - H.Q\}.\beta = G$. We can written $W\beta = G$ or $[W; G]$;$W = [w_{pq}]_{(n+1)\times(m+1)}$.
Where, $W = X.Z^T - H.Q$. Therefore, the approximate solution of Eq.(1) is calculated by

$$u_{N,M}(x,y) = X(x,y).Z^T.\beta$$

4 ERROR ANALYSIS

We have, $\forall(x,y) = (x_k, y_l) \in [0,1] \times [0,1]$; $k, l = 0,1,2, \ldots$:

$$E(x_k, y_l) = \left| u(x,y) - \int_a^b \int_c^d K(x,y,t,s)dsdt - g(x,y) \right| \cong 0$$

Where, $E(x_k, y_l) \leq 10^{-(n_k + m_l)}$($n_k$, m_l are positive integers). If we can determine[2]

$$\{Max_{k,l} 10^{-(n_k + m_l)}\} = 10^{-\varepsilon}$$

When M and N are large enough, the error can be estimated by the following function

$$E_{N,M}(x,y) = u(x,y) - \int_a^b \int_c^d K(x,y,t,s)dsdt - g(x,y)$$

In this case, if $E_{N,M}(x,y) \to 0$, then the error will decrease.

5 ILLUSTRATIVE EXAMPLES

In this section, We examine two examples for $N, M = 4, 8, 16$ with error values, exact solution, and two-dimensional modication of hat functions [3] and Taylor matrix [4]methods in tables. Estimation of absolute errors in tables is the values of $E_{N,M}(x,y) = \left|u(x,y) - u_{N,M}(x,y)\right|$ in the selected points.

EXAMPLE 5.1. Consider two-dimensional linear Fredholm integral equation below

$$u(x,y) = x\cos(y) - \frac{1}{6}\sin(1)\left(3 + \sin(1)\right) + \int_0^1 (\sin(t)s + 1)u(t,s)dsdt \ ; \ \ 0 \le (x,y) \le 1 \tag{16}$$

The exact solution is $x\cos(y)$.

(x,y)	**N=4**	**Error$_{4,4}$**	**Error$_{4,4}$[1]**	**N=8**	**Error$_{8,8}$**	**Error$_{8,8}$[1]**	**Exact**
(0.500,0.500)	0.453981	1.5e-02	0.6575e-03	0.451992	1.3e-02	0.1603e-06	0.438791
(0.250,0.250)	0.257418	1.5e-02	0.6467e-03	0.255429	1.3e-02	0.1602e-06	0.242228
(0.125,0.125)	0.139216	1.5e-02	0.6467e-03	0.137225	1.3e-02	0.1602e-06	0.124025
(0.063,0.063)	0.077568	1.5e-02	0.6467e-03	0.075579	1.3e-02	0.1602e-06	0.062378
(0.031,0.031)	0.046425	1.5e-02	0.6467e-03	0.044435	1.3e-02	0.1602e-06	0.031235
(0.016,0.016)	0.030813	1.5e-02	0.6467e-03	0.028824	1.3e-02	0.1602e-06	0.015623

Table 1 Compare the values of exact and approximate solutions Eq.(16)

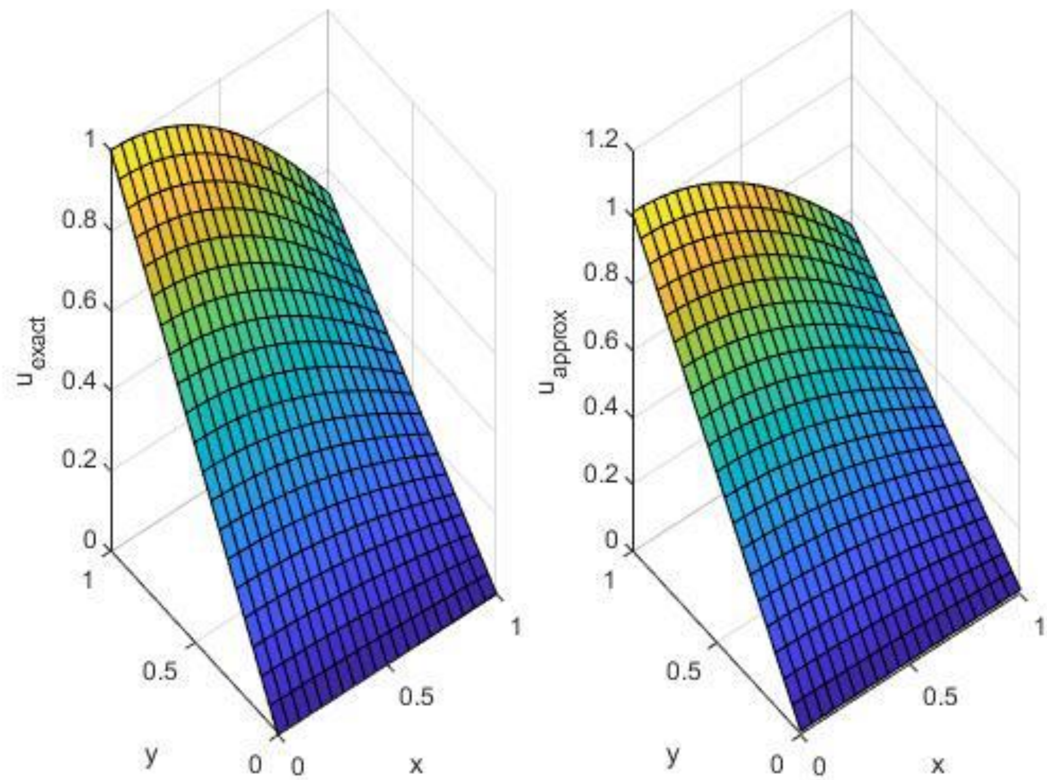

Figure 1: Exact and approximate solutions Eq.(16) for N=8

Example 5.2. Consider the following two-dimensional linear Fredholm integral equation

$$u(x,y) = g(x,y) + \int_0^1 \int_0^1 (x^2 + y + s^2 + t)u(t,s)dsdt \tag{17}$$

Where $g(x,y) = \frac{x^2}{3} + y^2 - \frac{2}{3}y - \frac{131}{180}$, and exact solution is $u(x,y) = x^2 + y^2$.

Table 2 Compare the values of exact and approximate solutions Eq.(17)

(x,y)	**N=8**	**Error$_{8,8}$**	**Error$_{8,8}$[2]**	**N=16**	**Error$_{16,16}$**	**Error$_{16,16}$[2]**	**Exact**
(0.500,0.500)	0.500000	1.2e-15	1.8e-14	0.500000	1.6e-07	1.8e-07	0.500000
(0.250,0.250)	0.125000	4.2e-16	6.3e-15	0.125000	7.0e-08	8.2e-08	0.125000
(0.125,0.125)	0.31250	2.1e-16	2.4e-15	0.31250	3.4-e08	4.0e-08	0.31250
(0.063,0.063)	0.007813	1.4e-16	2.1e-15	0.007813	1.5e-08	2.1e-08	0.007813
(0.031,0.031)	0.001953	1.2e-16	1.8e-14	0.001953	6.3e-09	7.4e-09	0.001953
(0.016,0.016)	0.000488	1.1e-16	1.7e-15	0.000488	1.7e-09	1.9e-09	0.000488

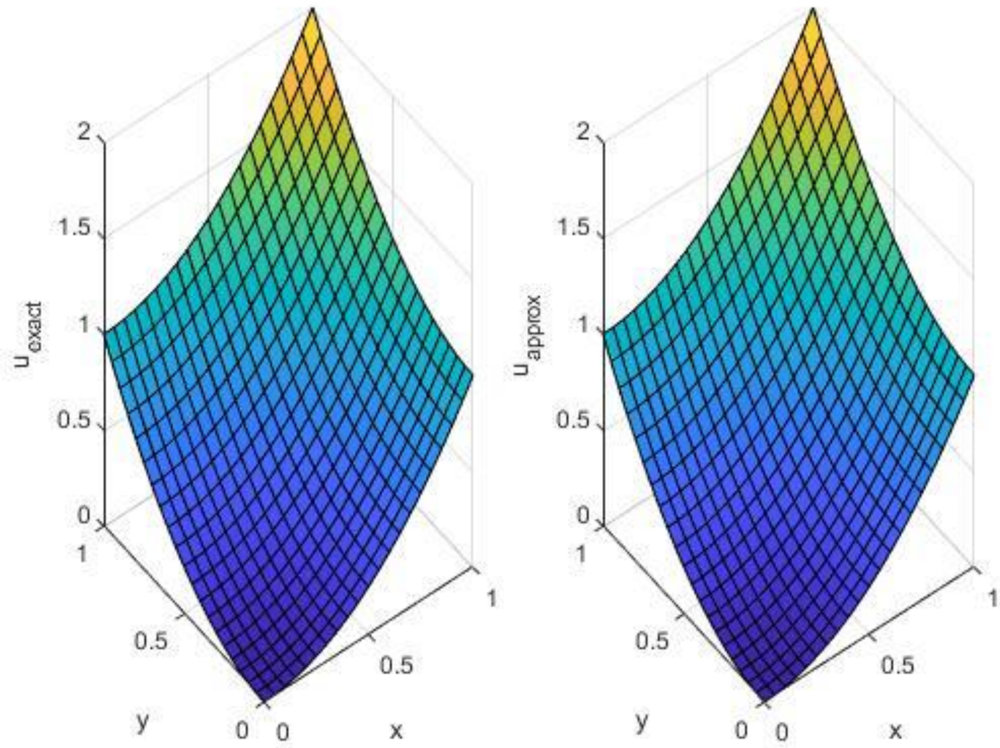

Figure 2: Exact and approximate solutions Eq.(17) for N=16

6 CONCLUSION

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In this paper, Boubaker collocation methods were used to the numerical approach of the two-dimensional linear integral equations of the second kind. The method has the best advantage, when the known functions in the equation can be extended to the terminate Boubaker series. Another notable advantage of the method is that the solution of the Boubaker polynomial cofficients is calculated very easy with using the computer programs. To get the solution of the best approximationin the equation , more sentences need to be used to expand thr Boubaker functions. That's means,the N terminate bound must be choosen large enough. In addition, an intersting feature of this method is nding analytical solutions. If the equation is a polynomial function, then it will have an exact solution. The results of the tables and _gures obtained from MATLAB software show the accuracy and efficiency of the method. We anticipate that the Boubaker collocation method would be promising for a detailed study of analytical solution for two-dimensional linear Fredholm integral equations.

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