5th International Conference on
Combinatorics, Cryptography, Computer Science and Computing
March. 4-5, 2021 - Ghaem Technology Institute of Higher Education

The signed total double Roman domatic number of a graph

Mina Valinavaz[1]
Department of Mathematics, Azarbaijan Shahid Madani University Tabriz, I.R. Iran

Abstract

A *signed total double Roman dominating function* ($STDRDF$) on a graph G is a function $f : V \to \{-1, 1, 2, 3\}$ such that $\sum_{u \in N(u)} f(u) \geq 1$ for every $v \in V(G)$, and every vertex $u \in V(G)$ for which $f(u) = -1$, is adjacent to at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, and if $f(u) = 1$, then vertex u must have at least one neighbor w with $f(w) \geq 2$. The weight of a signed total double Roman dominating function f is the value, $f(V(G)) = \sum_{u \in V(G)} f(u)$.The *signed total double Roman domination number* $\gamma_{stdR}(G)$ of G is the minimum weight of a $STDRDF$ of G. A set $\{f_1, f_2, ..., f_d\}$ of distinct signed total double Roman dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(G)$, is called a signed total double Roman dominating family (of functions) on G. The maximum number of functions in a signed total double Roman dominating family on G is the signed total double Roman domatic number of G, denoted by $d_{stdR}(G)$. In this paper we initiate the study of signed total double Roman domatic number in graphs and we present some sharp bounds for $d_{stdR}(G)$. In addition, we determine the signed total double Roman domatic number of some graphs. signed total double Roman dominating function, signed total double Roman domination number, signed total double Roman domatic number 05C69

1 Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *order* $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* of v is the set $N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = |N(v)|$. C_n for the cycle of length n, and P_n for the path of order n. A tree is a connected acyclic graph and a cactus is a connected graph in which every block is an edge or a cycle. For notation and graph theory terminology in general we follow [5]. A *signed total double Roman dominating function* ($STDRDF$) on a graph G is defined in [1]as a function $f : V \to \{-1, 1, 2, 3\}$ such that $\sum_{u \in N(u)} f(u) \geq 1$ for every $v \in V(G)$, and every vertex $u \in V(G)$ for which $f(u) = -1$, is adjacent to at least two neighbors assigned 2 under f or one neighbor w with $f(w) = 3$, and if $f(u) = 1$, then vertex u must have at least one neighbor w with $f(w) \geq 2$. The weight of a signed double Roman dominating function f is the value, $f(V(G)) = \sum_{u \in V(G)} f(u)$.The *signed double Roman domination number* $\gamma_{sdR}(G)$ of G is the minimum weight of a $STDRDF$ of G. A signed total double Roman dominating function $f : V \to \{-1, 1, 2, 3\}$ can be represented by the ordered partition $(V_{-}1, V_1, V_2, V_3)$ of V, where $V_i = \{v \in V | f(v) = i\}$. In this representation , its

[1]speaker

weight is $\omega(f) = |V_1| + 2|V_2| + 3|V_3| - |V_{-1}|$.
A set $\{f_1, f_2, ..., f_d\}$ of distinct signed total double Roman dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V(G)$, is called a signed total double Roman dominating family (of functions) on G. The maximum number of functions in a signed total double Roman dominating family ($STDRDfamily$) on G is the signed total double Roman domatic number of G, denoted by $d_{stdR}(G)$

The signed total double Roman domatic number is well defined and

$$d_{stdR}(G) \geq 1$$

for all graphs G since the set consisting of the $STDRDF$ with constant value 1 forms an $STDRD$ family on G. If $G_1, G_2, ..., G_k$ are the connected components of G, then obviously $d_{stdR}(G) = min\{d_{stdR}(G_i)|1 \leq i \leq k\}$. Hence, only consider connected graphs.
Our purpose in this paper is to initiate the study of signed double Roman domatic number in graphs. We first study basic properties and bounds for the signed total double Roman domatic number of a graph. In addition, we determine the signed total double Roman domatic number of some classes of graphs.
We make use of the following results in this paper.[2, 4, 3]

Proposition 1.1. For $p \geq 3$, $\gamma_{stdR}(K_{p,p}) = 2$, unless $p = 4$ in which case $\gamma_{stdR}(K_{4,4}) = 4$.

Proposition 1.2. *[1]* If C_n is the cycle on $n \geq 3$ vertices, then

$$\gamma_{stdR}(C_n) = \begin{cases} 6 & \text{if} \quad \text{n=5} \\ n & \text{if} \quad \text{otherwise} \end{cases}$$

Proposition 1.3. *[1]* If P_n is the path on $n \geq 2$ vertices, then

$$\gamma_{stdR}(P_n) = \begin{cases} n+1 & \text{if} \quad \text{n} \in 2,3,5,6,9 \\ n & \text{if} \quad \text{otherwise} \end{cases}$$

Proposition 1.4. *For $\delta \geq 1$, if G is an δ-regular graph of order n, then $\gamma_{stdR}(G) \geq \frac{n}{\delta}$.*

For $n \geq 2$, $\gamma_{stdR}(K_n) = 3$, unless $n = 4$ in which $\gamma_{stdR}(K_4) = 4$.
For $n \geq 2$, $\gamma_{stdR}(G) \leq \frac{3n}{2}$, with equality if and only if $G \simeq kP_2$.

2 Properties of the signed total double Roman domatic number

In this section we present basic properties of $d_{stdR}(G)$ and sharp bounds on the signed double Roman domatic number of a graph.

Theorem 2.1. For every graph G,

$$d_{stdR}(G) \leq \delta(G).$$

Moreover, if $d_{stdR}(G) = \delta(G)$, then for each $STDRD$ family $\{f_1, f_2, ..., f_d\}$ on G with $d = d_{stdR}(G)$ and each vertex v of minimum degree, $\sum_{u\in N(u)} f_i(u) = 1$ for each function f_i and $\sum_{i=1}^{d} f_i(u) = 1$ for all $u \in N(u)$

Proof. It $d_{stdR}(G) = 1$, the result is immediate. Let now $d_{stdR}(G) \geq 2$ and let $\{f_1, ..., f_d\}$ be an $STDRD$ family on G such that $d = d_{stdR}(G)$. Assume that v is a vertex of minimum degree $\delta(G)$. We have

$$d \leq \sum_{i=1}^{d} \sum_{u\in N(v)} f_i(u) = \sum_{u\in N(v)} \sum_{i=1}^{d} f_i(u) \leq \sum_{u\in N(v)} 1 = \delta(G).$$

Thus $d_{stdR}(G) \leq \delta(G)$.

If $d_{stdR}(G) = \delta$, then the two inequalities occurring in the proof become equalities. Hence for the $STDRD$ family $\{f_1, f_2, ..., f_d\}$ on G and for each vertex v of minimum degree, $\sum_{u\in N(v)} f_i(u) = 1$ for each function f_i and $\sum_{i=1}^{d} f_i(u) = 1$ for all $u \in N(v)$. □

The next results are immediate consequences of proposition and Theorem 2.1.

Corollary 2.2. *For any tree T of $n \geq 2$, $d_{stdR}(T) = 2$.*

Theorem 2.3. If G is a graph of order $n \geq 2$ with $\delta \geq 1$, then

$$\gamma_{stdR}(G).d_{sdR}(G) \leq \frac{3n}{2}.$$

Moreover, if $\gamma_{stdR}(G).d_{stdR}(G) = \frac{3n}{2}$, then for each $STDRD$ family $\{f_1, f_2, ..., f_d\}$ on G with $d = d_{stdR}(G)$, each function f_i is a $\gamma_{stdR}(G)$-function and $\sum_{i=1}^{d} f_i(v) = 1$ for all $v \in V$.

Proof. Let $\{f_1, f_2, ..., f_d\}$ be an $STDRD$ family on G such that $d = d_{stdR}(G)$ and let $v \in V$. Then

$$\begin{array}{rcl} \frac{3d}{2}.\, d_{stdR}(G) & = & \frac{3}{2}\sum_{i=1}^{d} \gamma_{stdR}(G) \\ & \leq & \frac{3}{2}\sum_{i=1}^{d}\sum_{v\in V} f_i(v) \\ & = & \frac{3}{2}\sum_{v\in V}\sum_{i=1}^{d} f_i(v) \\ & \leq & \frac{3}{2}\sum_{v\in V} 1 \\ & = & \frac{3n}{2} \end{array}.$$

If $\gamma_{stdR}(G).d_{stdR}(G) = \frac{3n}{2}$, then the two inequalities occurring in the proof become equalities. Hence for the $STDRD$ family $\{f_1, f_2, ..., f_d\}$ on G and for each i, $\sum_{v\in V} f_i(v) = \gamma_{stdR}(G)$. Thus each function f_i is a $\gamma_{stdR}(G)$-function, and $\sum_{i=1}^{d} f_i(v) = 1$ for all $v \in V$. □

Proposition 2.4. *If $p \geq 3$ and $p \neq 4$ is an integer, then $d_{stdR}(K_{p,p}) = p$.*

Corollary 2.5. If C_n is a cycle of length n, then $d_{stdR}(C_n) = 1$

Proof. Now let $P \neq 4$ and let $A = \{u_0, u_1, ..., u_{p-1}\}, B = \{v_0, v_1, ..., v_{p-1}\}$ be the bipartition of $K_{p,p}$.

Case 1. Assume that $n \equiv 0(mod\ 3)$. Define the functions $f_1, f_2, ..., f_p$ as follows:
$f_1(u_0) = f_1(v_0) = 3$, $f_1(u_i) = f_1(v_i) = 2$ if $1 \leq i \leq \frac{p}{3} - 1$ and $f_1(u_i) = f_1(v_i) = -1$ if $\frac{p}{3} \leq i \leq p$, in addition define $f_j(u_i) = f_j(v_i) = f_{j-1}(u_{i+j-1})$ for $0 \leq i \leq p-1, 2 \leq j \leq p$.
Where the indices are taken modulo p. It is straight forward to verify that f_i is a signed total double Roman dominating function an $K_{p,p}$ for $1 \leq i \leq p$ and $\{f_1, f_2, ..., f_p\}$ is a signed total double Roman

dominating family on $K_{p,p}$. Therefore $d_{stdR}(K_{p,p}) \geq p$. Therefore $d_{stdR}(K_{p,p}) = p$ by theorem 2.1.
case 2. Assume that $n \equiv 1(mod\ 3)$. Define the functions $f_1, f_2, ..., f_p$ as follows:
$f_1(u_0) = f_1(v_0) = f_1(u_1) = f_1(v_1) = 3$, $f_1(u_i) = f_1(v_i) = 2$ if $2 \leq i \leq \frac{p-1}{3} - 1$ and $f_1(u_i) = f_1(v_i) = -1$ if $\frac{p-1}{3} \leq i \leq p$, in addition define $f_j(u_i) = f_j(v_i) = f_{j-1}(u_{i+j-1})$ for $0 \leq i \leq p-1, 2 \leq j \leq p$.
Where the indices are taken modulo p. It is straight forward to verify that f_i is a signed total double Roman dominating function an $K_{p,p}$ for $1 \leq i \leq p$ and $\{f_1, f_2, ..., f_p\}$ is a signed total double Roman dominating family on $K_{p,p}$. Therefore $d_{stdR}(K_{p,p}) \geq p$. Therefore $d_{stdR}(K_{p,p}) = p$ by theorem 2.1.
case 3. Assume that $n \equiv 2(mod\ 3)$. Define the functions $f_1, f_2, ..., f_p$ as follows:
$f_1(u_i) = f_1(v_i) = 2$ if $0 \leq i \leq \frac{p+1}{3} - 1$ and $f_1(u_i) = f_1(v_i) = -1$ if $\frac{p+1}{3} \leq i \leq p$, in addition define $f_j(u_i) = f_j(v_i) = f_{j-1}(u_{i+j-1})$ for $0 \leq i \leq p-1, 2 \leq j \leq p$.
Where the indices are taken modulo p. It is straight forward to verify that f_i is a signed total double Roman dominating function an $K_{p,p}$ for $1 \leq i \leq p$ and $\{f_1, f_2, ..., f_p\}$ is a signed total double Roman dominating family on $K_{p,p}$. Therefore $d_{stdR}(K_{p,p}) \geq p$. Therefore $d_{stdR}(K_{p,p}) = p$ by theorem 2.1. $\square$

Proposition 2.6. *Let G be a δ-regular graph of order n such that $\delta \geq 1$ then $d_{stdR}(G) \leq \delta$*

Proof. Let $\{f_1, f_2, ..., f_d\}$ be a $STDRD$ family on G such that $d = d_{stdR}(G)$, and let $\delta \geq 1$ and $n = p\delta + r$ with integers $p \geq 1$ and $0 \leq r \leq \delta$. It follows that

$$\sum_{i=1}^{d} w(f_i) = \sum_{i=1}^{d} \sum_{v \in V} f_i(v) \leq \sum_{v \in V} 1 = n$$

We deduce from proposition 1.4 that $w(f_i) \geq \gamma_{stdR}(G) \geq p + \frac{r}{\delta}$ for each $i \in \{1, 2, ..., d\}$. Suppose to the contrary that $d \geq \delta + 1$. Then we obtain

$$\sum_{i=1}^{d} w(f_i) \geq pd + \frac{dr}{\delta} \geq p(\delta + 1) + \frac{(\delta+1)r}{\delta} > n.$$

Thus $d \leq \delta$. $\square$

Theorem 2.7. *Let G be a graph of order n with $\delta \geq 1$. Then*

$$\gamma_{stdR}(G) + d_{stdR}(G) \leq \frac{3n}{2} + 1$$

with equality if and only if the components of $G \simeq kP_2$ for $k \geq 1$.

Proof. It follows from theorem2.3 that

$$\gamma_{stdR}(G) + d_{stdR}(G) \leq \frac{3n}{2d_{stdR}(G)} + d_{stdR}(G).$$

According to theorem 2.1, we have $1 \leq d_{sdR}(G) \leq n-1$. Using these bounds, and the fact the function $g(x) = x + \frac{\frac{3n}{2}}{x}$ is decreasing for $1 \leq x \leq \sqrt{\frac{3n}{2}}$ and increasing for $\sqrt{\frac{3n}{2}} \leq x \leq n-1$, we obtain

$$\gamma_{stdR}(G) + d_{stdR}(G) \leq \frac{3n}{2d_{stdR}(G)} + d_{stdR}(G) \leq max\{\frac{3n}{2} + 1, \frac{3n}{2(n-1)} + (n-1)\} = \frac{3n}{2} + 1.$$

and the desired bound is proved. If $G \simeq kP_2$ then it follows from proposition 1 implies that $\gamma_{stdR}(G) = \frac{3n}{2}$ and by theorem 2.1, $d_{stdR}(G) = 1$ and so $\gamma_{stdR}(G) + d_{stdR}(G) = \frac{3n}{2} + 1$.
Conversely, assume that $\gamma_{stdR}(G) + d_{stdR}(G) = \frac{3n}{2} + 1$. The inequality above lead to

$$\frac{3n}{2} + 1 = \gamma_{stdR}(G) + d_{stdR}(G) \leq \frac{3n}{2d_{stdR}(G)} + d_{stdR}(G) \leq max\{\frac{3n}{2} + 1, \frac{3n}{2(n-1)} + (n-1)\} = \frac{3n}{2} + 1$$

This implies that $d_{stdR}(G) = 1$, thus $\gamma_{stdR}(G) = \frac{3n}{2}$, and it follow from proposition 1 that $G \simeq kP_2$. $\square$

References

[1] Ahangar, l.shahbazi, M., Sheikholeslami, S. M. *Signed total double Roman domination in graphs*, Discrete Applied Mathematics. **257**(2017): 1-11.

[2] Amjadi, J., Yang, H., Nazari-Moghaddam, S., Sheikholeslami, S. M., Shao, Z. *Signed double Roman k-domination in graphs*, Australasian J. Combinatorics **72** (2018): 82-105.

[3] Sheikholeslami, S. M., Volkmann.*The signed Roman domatic number of a graph*, A. Math.In. **40**(2012): 105-112.

[4] Volkmann, Lutz. *On the signed total Roman domination and domatic numbers of graphs*, Discrete Applied Mathematics **214** (2016): 179-186.

[5] West, D.B, *Introduction to Graph Theory*, Prentice-Hall,2000.

m.valinavaz@azaruniv.ac.ir