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New self-orthogonal designs from the group J_2

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ABSTRACT

In this paper, we construct new self-orthogonal 2-designs on 100 points with block stabilizer of order greater than four that admit the Janko group J_2 as an automorphism group. Also, the Janko group J_2 acts primitively on the points and transitively on the blocks of this design.

KEYWORDS: t-design, Permutation representation, Self-orthogonal design, Automorphism group.

1 INTRODUCTION

One of the interesting problems in design theory is the problem of constructing simple t−designs which has been studied by many authors. There are several major approaches to the problem. There are several computational methods for constructing t-designs. For instance, constructing t−designs by using prescribed automorphism groups, such as, Kramer and Mesner method or constructing t−designs via recursive construction methods. For the computational method used to construct some t-designs, we refer the reader to [1, 3, 6, 9, 10].

Assume that $D=(V,B)$ is a $t-(v,k,\lambda)$ design with an automorphism group G such that G acts transitively on blocks. Now assume that H is the stabilizer of a block. Then, it is well-known that this block is a union of some orbits of the group H. By using this fact, in the current paper, $t-(v,k,\lambda)$ design with a prescribed group of automorphisms is constructed. It is well known that when G is a t-homogeneous group, then the action of G on each orbit forms a t-design. Therefore, in this method the problem is that how one could assemble orbits to make a t-design.

Now, we consider the action of the Janko group J_2 on 100 points. New design, as the main result of this paper, which has been computed with the aid of GAP [11] is presented in the following proposition.

Proposition 1. There is one class of $2-(100, 22, 4032)$ self-orthogonal design with the automorphism group J_2.

2 NOTATION AND PRELIMINARIES

We assume that the reader is familiar with the basic facts of group theory and design theory. We refer the reader to [2, 4] for relevant background reading in design theory and to [5, 12] for relevant background reading in group theory. So, we introduce the notation and preliminaries used in the construction process, briefly.

Let t, v, k be non-negative integers such that $v \geq k \geq t \geq 0$. A $t-(v, k, \lambda)$ design is a set of v points together with a collection of k-element subsets called blocks such that every t-subset of points is contained in exactly λ blocks. In this case, the design is shown by $D=(V,B)$ such that V is the set of points and B is the set of blocks. The design is simple if B is a set, that is, contains no repeated blocks. The complement of a design (V,B) is $(V,\overline{B})$, where $\overline{B}=\{V \setminus b : b \in B\}$. In fact, the complement of a $t-(v, k, \lambda)$ design is a $t-(v, k, \lambda^{*})$ design, where

$$\lambda^{*}=\frac{\lambda\binom{v-k}{t}}{\binom{k}{t}}.$$

A design is called self-complement when the design and its complement are the same. The residual design of a t-design is obtained by removing a point from the point set as well as removing the blocks that contain the removed point from the collection of blocks. If the original t-design has parameters t, v, k, and λ then the residual t-design is a $(t-1)-(v, k, \lambda^{*})$ design, where 2 - (100, 22, 4032)

$$\lambda^{*}=\frac{\lambda(v-k)}{k-t+1}.$$

Let v be a point, and B be the collection of blocks in a t-design that contain the point v**.** The derived design consists of the point set $V \setminus v$ and the blocks B^{*}, obtained from removing v from each block in B**.** If the original t-design has parameters t, v, k, and λ then the derived t-design is a $(t-1)-(v-1, k-1, \lambda)$ design. If all blocks of D are distinct, then it is called simple. Throughout this paper, we are concerned only with simple designs. A t-design is called self-orthogonal if the block intersection numbers, which are the cardinalities of the intersections of any two distinct blocks, have the same parity as the block size k [13].
An automorphism of a design $D=(V,B)$ is a permutation on V which sends blocks to blocks. The set of all automorphisms forms a group called the full automorphism group of D and is denoted by $\mathrm{Aut}(D)$ .A subgroup G of $\mathrm{Aut}(D)$ is block-transitive if G acts transitively on the set of blocks of D. In this case, D is called block-transitive. Also, Point-transitivity is defined similarly.

3 THE CONSTRUCTION

Before the beginning of our proposition proof, let us mention some property of the sporadic simple group J_2. The Janko simple group J_2 has order 604800. The Janko group J_2 has nine classes of maximal subgroups. Consider the permutation representation of J_2 on 100 right cosets of the maximal subgroup $U_3(3)$. The group J_2 is 1-transitive and primitive in this representation [7]. In this section, we consider the permutation representation of J_2 on 100 points with generators,

$$\alpha=(1,84)(2,20)(3,48)(4,56)(5,82)(6,67)(7,55)(8,41)(9,35)(10,40)(11,78)(12,100)(13,49)$$
$$(14,37)(15,94)(16,76)(17,19)(18,44)(21,34)(22,85)(23,92)(24,57)(25,75)(26,28)(27,64)$$
$$(29,90)(30,97)(31,38)(32,68)(33,69)(36,53)(39,61)(42,73)(43,91)(45,86)(46,81)(47,89)$$
$$(50,93)(51,96)(52,72)(54,74)(58,99)(59,95)(60,63)(62,83)(65,70)(66,88)(71,87)$$
$$(77,98)(79,80),$$

$$\beta = (1,80,22)(2,9,11)(3,53,87)(4,23,78)(5,51,18)(6,37,24)(8,27,60)(10,62,47)(12,65,31)$$
$$(13,64,19)(14,61,52)(15,98,25)(16,73,32)(17,39,33)(20,97,58)(21,96,67)(26,93,99)$$
$$(28,57,35)(29,71,55)(30,69,45)(34,86,82)(38,59,94)(40,43,91)(42,68,44)(46,85,89)$$
$$(48,76,90)(49,92,77)(50,66,88)(54,95,56)(63,74,72)(70,81,75)(79,100,83).$$

The group J_2 up to conjugation has 146 subgroups. It suffices to find the non-conjugate subgroups of J_2 because we need only one representative of each k-subset under the action of J_2. If H_1 and H_2 are conjugate subgroups of J_2, then H_1 fixes the k-subset K if and only if H_2 fixes K^{α} (for some $\alpha \in J_2$ such that $H_2 = H_1^{\alpha}$). We use the GAP command ConjugacyClassesSubgroups(G) to find non-conjugate subgroups.

Let H be a representative of a conjugacy class of subgroups of J_2 Now, consider a union of orbits of H, say X. We construct the orbit of X under the action of J_2. Since J_2 acts 1-transitively, the orbit of X under the action of J_2 forms a 1-design. We used this method for each union of orbits of H and we will be interested when the constructed designs are 2-design. We applied this method for each conjugacy class of subgroups of J_2 for constructing designs. By the above discussion, consider $X = \{1, 9, 11, 13, 27, 33, 38, 42, 44, 46, 49, 53, 54, 55, 57, 66, 67, 72, 77, 79, 92, 95\}$.
The orbit of X under the action of J_2 forms a self-orthogonal $2-(100, 22, 4032)$ design. This design has 86400 blocks. The stabilizer of the block is a subgroup of order 7 with orbit lengths $1^2, 7^{14}$. The full automorphism group of this design is isomorphic to J_2. The block intersection numbers of this design are $0^{339}, 2^{10717}, 4^{36743}, 6^{30240}, 8^{7681}, 10^{651}, 12^{21}, 14^{7}$.
The group J_2 acts primitively on the points and transitively on the blocks of this design. This is block transitive self-orthogonal 2-designs on 100 points with block stabilizer of order greater than four that admit the group J_2 as an automorphism group. The results are concluded by computations with the aid of GAP is based on the procedure described. This completes the proof of Proposition 1.

We consider the derived and residual designs of this design. The derived and residual designs of 2-designs are 1 - (99, 21, 4032) and 1 - (100, 22, 14976)

4 CONCLUSION

According to Proposition 1, up to complementary there exist block-transitive simple $2-(100, 22, 4032)$ designs with automorphism group J_2. We study the properties of one member for a class of isomorphic designs. The automorphism group acts primitively on the points and transitively on the blocks of this design.

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