Richardson extrapolarion on numerical approximations of combination of Laplacian and biharmonic operator and its application in edge detection

Hamid Mesgarani , Mohsen Shahrezaee

Department of science University of Shahid Rajaie ,Tehran,Iran;

Imam hussain university , shahid babaee highway, Tehran,Iran

hmesgarani@sru.ac.ir ; Mshahrezaee@bmn.ir

Parisa Parmour

Phd student ,shahid rajaie university

parisa.parmour@gmail.com

ABSTRACT

This paper applies numerical approximations of combination of Laplacian and biharmonic operator for edge detection. The Laplacian of an image combine the second order derivatives to detect edge of an image. Numerical approximation are computed by finite difference schemes of $O(h^4)$ then we apply Richardson's extrapolation technique in order to achieve better approximation of $O(h^6)$ for edge detection and PSNR values are presented to show improvement in results of edge detection for Richardson extrapolation. Finally the experimental results of extrapolation edge detector for comparison with Laplacian and Canny edge detectors are presented.

KEYWORDS: numerical approximation, Laplacian operator, biharmonic operator ,edge detection, finite difference , Richardson extrapolation, PSNR values.

1 INTRODUCTION

Digital image processing plays an important role in modern culture for examples: medical field , Biology , botany, zoology, cell biology, microbiology, biochemistry, Biometrics, Environmental sciences, Robotics, Professional sport, Astronomy and etc. [1]

this paper focused on edge detection of an digital image. An edge is a rapid transition between dark and light areas of an image. The goal of edge detection is to identify locations in an image where such transitions are particularly strong. Strong edges indicate significant visual features of an image such as boundaries, change in texture ,change in scene depth and so on .

Edge detection techniques are based on derivatives of an image. Since digital images are not continuous but discrete, the approximation of a digital derivatives must be computed numerically.

There is a many papers on the subject of edge detection . an article by prewitt [2] gives a perspective on early work in this field. The Laplacian of Gaussian filter was first described by Marr and Hildreth [3]. Furture information of a Canny edge detector can be found in the original paper by Canny[4]. For applying Laplace operator for edge detection see also[5,6,7,8].

This paper is organized as follows. section2 describes numerical approximation of combination of Laplacian and biharmonic operator of $O(h^4)$. Section 3 presents the experimental results of applying numerical approximation of section 2 on test images . Section 4 describes Richardson extrapolation technique on numerical approximation which is applied in section 3, in order to obtain better approximation of $O(h^6)$ and PSNR values is presented in order to show improvement in edge detection for Richardson extrapolation rather than approximations of $O(h^4)$. In Section 5 exprimental results of applying extrapolation technique and camparison with Laplacian and Canny edge detectors are presented. Finally the conclusions are given is section 6.

2 NUMERICAL APPROXIMATIONS FOR EDGE DETECTION

Images are functions of two variables . it is usual to digitize the values of the image function , $u(x, y)$,in addition to its spatial coordinates. This process of quantisation involves replacing a continuously varying $u(x, y)$ with a discrete set of quantisation levels. The more levels we use, the better the approximation. Let $u_{i,j}$ represent an approximation to $u(x_i, y_j)$ and Laplacian of function u is defined with relation (1):

$$\nabla^2 u(x, y) = u_{xx}(x, y) + u_{yy}(x, y) \tag{1}$$

approximation of Laplacian operator is applied for edge detection in image processing [9,10].

In order to discretize (1) , usually both the x and y derivatives of u are replaced by centered finite differences of $O(h^2)$,which gives:

$$\nabla^2 u_{ij} = \frac{1}{(\Delta x)^2}\left(u_{i-1,j} - 2u_{ij} + u_{i+1,j}\right) + \frac{1}{(\Delta y)^2}\left(u_{i,j-1} - 2u_{ij} + u_{i,j+1}\right) \tag{2}$$

where $\Delta x = \Delta y = h.$ Then:

$$\nabla_5^2 u_{i,j} = \frac{1}{h^2}\left(u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j}\right) + O(h^2) \tag{3}$$

thus the 5.point convolution matrix for the approximation of Laplacian of $O(h^2)$ is computed with (4) and frequently is used as Laplacian operator for edge detection in most of image processing books, for example [7,11]:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{4}$$

Now by using general expression for the Taylor series in two variables for $u(x, y)$ about a point (x_i, y_j) for $u_{i-1,j}, u_{i+1,j}, u_{i,j-1}, u_{i,j+1}, u_{i-1,j-1}, u_{i-1,j+1}, u_{i+1,j-1}, u_{i+1,j+1}$, 9.point stencil for approximate $\nabla^2 u + \frac{h^2}{12}\nabla^2(\nabla^2 u)$ is obtained by relation (5):

$$\begin{aligned} &\frac{1}{6h^2}\left(4u_{i-1,j} + 4u_{i+1,j} + 4u_{i,j-1} + 4u_{i,j+1} + u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1} + u_{i+1,j+1} - 20u_{i,j}\right) \\ &= \left((u_{xx})_{i,j} + (u_{yy})_{i,j} + \frac{h^2}{12}(u_{xxxx})_{i,j} + \frac{h^2}{12}(u_{yyyy})_{i,j} + \frac{h^2}{6}(u_{xxyy})_{i,j}\right) + O(h^4) \\ &= \nabla^2 u_{i,j} + \frac{h^2}{12}\left(\nabla^4 u_{i,j}\right) + O(h^4) \end{aligned} \tag{5}$$

Where $\nabla^4 u = \nabla^2(\nabla^2 u)$ is biharmonic Operator.

In [10] this nine point stencil is presented and in [12] this approximation used for solving second order nonlinear elliptic partial differential equations and now we apply this approximation for calculating convolution matrix of $O(h^4)$ for edge detection of an image. convolution matrix of (5) is defined by (6):

$$B = \begin{bmatrix} 1 & 4 & 1 \\ 4 & -20 & 4 \\ 1 & 4 & 1 \end{bmatrix} \tag{6}$$

3 EXPERIMENTAL RESULTS OF NUMERICAL APPROXIMATION

In this paper we use spatial domain techniques with regional processing and the most important regional processing technique is known as convolution.Convolution employs a rectangular grid of coefficients, known as kernel , to determine how values of the particular pixel and values of its neighborhood elements affect the image matrix and form the result matrix for edge detection .Suppose that image matrix U is a $m \times n$ matrix where elements of a matrix $u_{i,j}$ are pixel values of image:

$$U = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \dots & \vdots \\ u_{m1} & u_{m2} & \dots & u_{mn} \end{bmatrix} \qquad (7)$$

With multiplication (componentwise) convolution kernels (convolution matrices) (6) to pixel values of image matrix (7) ,(that is $B * U$), then we have following result for edge detection. In this paper, if the kernel extends beyond the source image when the central particular pixel value in image matrix is $u_{i,j}$ then the output is defined as $u_{i,j}$. Infact this way ,copies the input to the convolved output at the boundaries. For test images In figure (1) , we apply nine point stencil (5) with convolution matrix (6) for edge detection.The result of edge detection are shown in figure (2).

Figure (1): test images

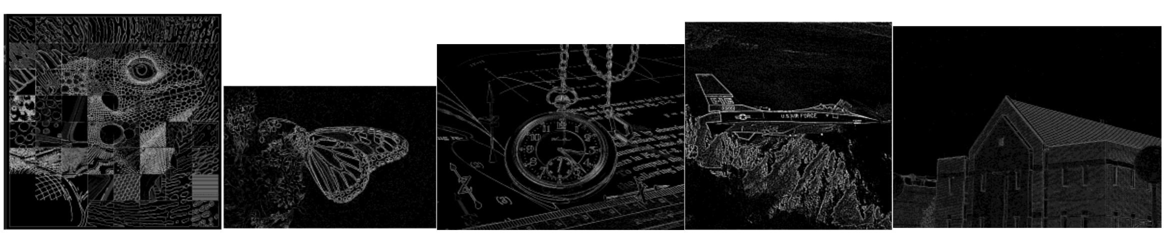


Figure (2) :the results of nine point stencil (5) for edge detection

4 RICHARDSON EXTRAPOLATION TECHNIQUE AND ITS APPLICATION IN EDGE DETECTION

The most immediate application of Richardson extrapolation is to numerical differentiation. This topic is considered in almost all books on numerical analysis ,for example see[13,8].

Let $D_k\left(\frac{h}{2}\right)$ and $D_k(h)$ be two approximations of $O\left(h^{2k}\right)$ for derivative of two variable functions , then an improved approximation has the form:

$$D_{k+1}(h)=\frac{4^k D_k\left(\frac{h}{2}\right)-D_k(h)}{4^k-1}+O\left(h^{2k+2}\right) \tag{8}$$

In section 2 we introduce the approximation of digital derivatives of $O\left(h^4\right)$ and in this section we use Richardson extrapolation technique and we get approximation of digital derivatives of $O\left(h^6\right)$.

Suppose that $u_{i,j}$ represent an approximation for $u\left(x_i, y_j\right)$, infact $u_{i,j}$ is pixel value of image and $\bar{u}_{i,j}$ is corresponding pixel value with $u_{i,j}$ in edge detection matrix(according to (5) and (6)). Then :

$$\bar{u}^1_{i,j}=4u_{i-1,j}+4u_{i+1,j}+4u_{i,j-1}+4u_{i,j+1}+u_{i-1,j-1}+u_{i-1,j+1}+u_{i+1,j-1}+u_{i+1,j+1}-20u_{i,j} \tag{9}$$

first we approximate derivatives with adjacent pixels of $u_{i,j}$ by relation (9) , and second we approximate derivatives by following relation (10) :

$$\bar{u}^2_{i,j}=4u_{i-2,j}+4u_{i,j-2}+4u_{i,j+2}+4u_{i+2,j}-20u_{i,j}+u_{i-2,j-2}+u_{i-2,j+2}+u_{i+2,j-2}+u_{i+2,j+2} \tag{10}$$

And third by applying Richardson extrapolation for (9) and (10) we have:

$$\bar{u}_{i,j}=\frac{4^2\bar{u}^1_{i,j}-\bar{u}^2_{i,j}}{4^2-1}+O\left(h^6\right) \tag{11}$$

In (9) , $\bar{u}^1_{i,j} = u_{i,j}$, for $i=1, i=m, j=1, j=n$ and in (10) , $\bar{u}^2_{i,j} = u_{i,j}$, for $i=1, i=2, i=m, i=m-1, j=1, j=2, j=n, j=n-1$ where m , n are dimensions of image. We consider these two approximation (9) and (10) as approximation by step length $\frac{h}{2}$ and h (respectively) and finally we apply Richardson extrapolation with relation (11) in order to achieve numerical approximation of $O(h^6)$ for edge detection for every pixel.

In order to compare results among relations (9) , (10) and (11) , we will use PSNR values of them. In [14,15] the values of PSNR is used to evaluate different edge detection methods. The more the PSNR values, the more exact the edge detection methods. Suppose $u_{i,j}$ and $\bar{u}_{i,j}$ denote the pixel values of the test images and that of the edge detectional image, respectively and m and n are the size of the images. PSNR values calculated from relation (12) :

$$PSNR = 10 \times \log_{10} \frac{255^2 \times m \times n}{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(u_{i,j} - \bar{u}_{i,j}\right)^2} \qquad (12)$$

The more the PSNR values, the more exact the edge detection methods. Table (1) presents the quantitative PSNR values for (9),(10) and (11) .

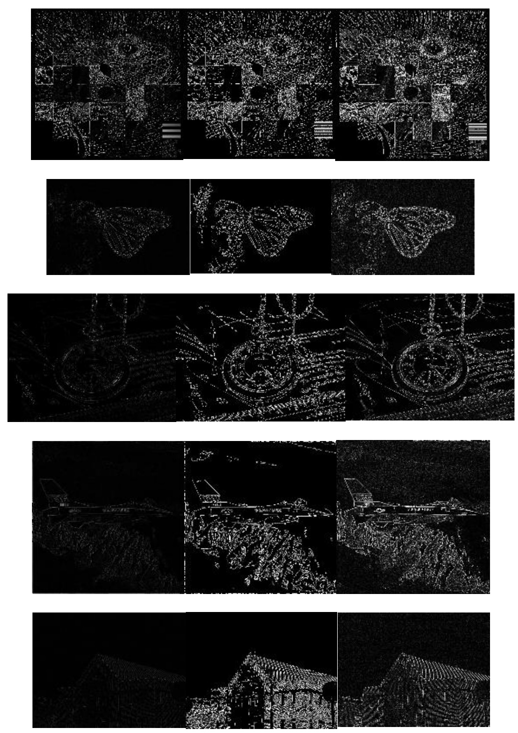
Test images	**(a)** $\bar{u}^2_{i,j}$	**(b)** $\bar{u}^1_{i,j}$	**(c)** $\bar{u}_{i,j}$ (Richardson extrapolation between $\bar{u}^1_{i,j}$ and $\bar{u}^2_{i,j}$)
frymire	37.054	39.75	**39.83**
monarch	46.29	50.78	**51.08**
watch	48.58	52.74	**52.99**
airplane	46.35	49.13	**49.24**
building	48.97	50.99	**51.08**

Table(1) PSNR values

The PSNR values in the table (1) verified that the approximation scheme presented by relation (11) is better than approximation scheme by relations (9) and (10) for edge detection and with comparing these two approximations and extrapolation method from table (1) we get that the extrapolation method is the best.

5 EXPERIMENTAL RESULT OF APPLYING RICHARDSON EXTRAPOLATION TECHNIQUE

In the following figures experimental results of applying Richardson extrapolation technique by relation (11) and Camparison with Laplacian and Canny edge detector are shown:



(a) (b) (c)

Figure (2): (a) results of Laplacian edge detector; (b) results of Canny edge detector ;(c) results of $\overline{u}_{i,j}$ by relation (11) (Richrdson extrapolation)

6 CONCLUSION

In this paper Richardson extrapolation method is applied to improve edge detection methods based on approximating of Laplacian and biharmonic operator . Proposed Richardson extrapolation method for edge detection in this paper has $O\left(h^{6}\right)$ and the proposed method have higher PSNR values.

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