Combinatorics and Fibonacci polynomials

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ABSTRACT

In mathematics, the Fibonacci polynomials are a polynomial sequence which can be considered as a generalization of the Fibonacci numbers. In this paper, we try to identify a specific type of Fibonacci polynomials using combinatorial methods.

KEYWORDS: Combinatorics, Fibonacci polynomials, Financial analysis, binomial expansion.

1 INTRODUCTION

In mathematics, the Fibonacci numbers, denoted by F_n , form a sequence, called the Fibonacci sequence, such that each number is the sum of the two preceding ones, starting from 0 and 1. That is,

$$F_{0,} = 0,\ F_{1,} = 1,\ F_n = F_{n-1} + F_{n-2} \text{ for } n > 1.$$

The beginning of the sequence is thus

$$\{0,1,1,2,3,5,8,13,21,34,55,89,144,\dots\}.$$

There is a strong relation between Fibonacci numbers and golden ratio, where

$$\frac{1+\sqrt{5}}{2} = \varphi \simeq 1.618\dots \quad ,\frac{1-\sqrt{5}}{2} \simeq -0.618\dots$$

Johannes Kepler proved that $\lim\limits_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi$.

In finance, Fibonacci retracement is a method of technical analysis for determining support and resistance levels. Fibonacci retracement are a popular technical analysis tool that help traders to identify future price movement. They are named after their use of the Fibonacci sequence. Fibonacci retracement is based on the idea that markets will retrace a predictable portion of a move, after which they will continue to move in the original direction.

According to Fibonacci, any nature-driven market, such as the financial market, is prone to make retracements that are either 0.618 (61.8%) or 0.382 (38.2%) of the distance a stock, currency, or index has moved. There are four Fibonacci tools have been able to utilize the golden ratio. They are calculated by locating the high and low of the market chart, then drawing five horizontal lines to indicate support and resistance areas. The first line is drawn at the highest point of the chart (100%), then the second to the fifth are drawn at 61.8%, 50%, 38.2% and 0% (lowest point on the chart) in that order. When a significant price movement happens, new support and resistance levels are established near these horizontal lines.

Fibonacci polynomials are a great important in Mathematics large classes of polynomial can be defined by Fibonacci-like recurrences relation and Fibonacci relation and Fibonacci numbers. Such polynomials were studied in 1883 by the Belgian Mathematician Eugene Charles Catalan and German Mathematician E. Jacobsthal. The polynomials $F_n(x)$ studied by Catalan are defined by

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ F_1(x) = 1, F_2(x) = x.$$

The Fibonacci polynomials studied by Jacobsthal were defined by

$$J_n(x) = J_{n-1}(x) + xJ_{n-2}(x),\ J_1(x) = 1, J_2(x) = 1.$$

We list the first members as following:

$F_1(x) = 1$	$J_1(x) = 1$
$F_2(x) = x$	$J_2(x) = 1$
$F_3(x) = x^2$	$J_3(x) = x + 1$
$F_4(x) = x^3 + 2x$	$J_4(x) = 1 + 2x$
$F_5(x) = x^4 + 3x^2 + 1$	$J_5(x) = x^2 + 3x + 1$
$F_6(x) = x^5 + 4x^3 + 3x$	$J_6(x) = 3x^2 + 4x^2 + 1$

For example, recently it is proved that

$$J_n(x) = (\sqrt{x})^{n-1} F_n\left(\frac{1}{\sqrt{x}}\right)$$

Combinatorial Method. In this manner, let $F_n(a,b)$ be the n-th Fibonacci polynomial defined by

$$F_n(a,b) = aF_{n-1}(a,b) + bF_{n-2}(a,b),\ F_1(a,b) = 1, F_0(a,b) = 0.$$

Where a, b are indeterminate. Let $L_n(a,b)$ be the Lucas polynomial defined

$$L_n(a,b) = aL_{n-1}(a,b) + bL_{n-2}(a,b),\ F_1(a,b) = \mathrm{a}, L_0(a,b) = 2.$$

The Fibonacci polynomial and Lucas polynomial are also given by the well-known formulas

$$F_n(a,b) = \sum_{k=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1-k}{k} a^{n-1-2k}\, b^k$$

and

$$L_n(a,b) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{n}{n-k} \binom{n-k}{n} a^{n-2k}\, b^k$$

when $a = b = 1$, $F_n(a,b)$ and $L_n(a,b)$ reduce to Fibonacci and Lucas sequence.

The Hessenberg matrix is defined as following

$$A_n = \begin{bmatrix} 2 & 1 & \cdots & 0 \\ 1 & 2 & & 0 \\ & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{bmatrix}$$

It is proved that $|A_n| = F_{n+2}$. For more details please see [1], [2], [3], [4], [5], [6].

In this article, we define a special type of Fibonacci polynomials as following:

$$F_{n+2}(x) = F_{n+1}(x)(2-4x) - F_n(x),\ F_0(x) = 1, F_1(x) = 3-4x.$$

and we prove

Main Theorem. With previous notation, we have :

$$F_n(x) = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} (1-x)^{n-k}\, x^k$$

2 MAIN RESULT

According to the introduction mentioned so far, we define a special type of Fibonacci polynomials as following:

$$F_{n+2}(x) = F_{n+1}(x)(2-4x) - F_n(x),\ F_0(x) = 1, F_1(x) = 3-4x.$$

Main Theorem. With previous notation, we have :

$$F_n(x) = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} (1-x)^{n-k}\, x^k$$

Proof. Notice that

$$\binom{2n+3}{2k+1} = \binom{2n+1}{2k+1} + 2\binom{2n+1}{2k} + \binom{2n+1}{2k-1}$$

So,

$$\sum_{k=0}^{n+1} (-1)^k \binom{2n+3}{2k+1} (1-x)^{n+1-k}\, x^k$$

$$= \sum_{k=0}^{n+1} (-1)^k \binom{2n+1}{2k+1} (1-x)^{n+1-k}\, x^k + \sum_{k=0}^{n+1} (-1)^k \binom{2n+1}{2k-1} (1-x)^{n+1-k}\, x^k$$

$$+ 2\sum_{k=0}^{n+1} (-1)^k \binom{2n+1}{2k} (1-x)^{n+1-k}\, x^k$$

$$= (1-2x)\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} (1-x)^{n-k}\, x^k +$$

$$2(1-x)\sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k} (1-x)^{n-k}\, x^k$$

On the other hand,

$$\binom{2n+1}{2k} = \binom{2n-1}{2k} + 2\binom{2n-1}{2k-1} + \binom{2n-1}{2k-2}$$

Thus, we conclude that

$$\sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k}(1-x)^{n-k}\,x^k$$
$$=\sum_{k=0}^{n}(-1)^k\binom{2n-1}{2k}(1-x)^{n+1-k}\,x^k+\sum_{k=0}^{n}(-1)^k\binom{2n-1}{2k-2}(1-x)^{n+1-k}\,x^k$$
$$+\,2\sum_{k=0}^{n}(-1)^k\binom{2n-1}{2k-1}(1-x)^{n+1-k}\,x^k$$
$$=(1-2x)\sum_{k=0}^{n-1}(-1)^k\binom{2n-1}{2k}(1-x)^{n-1-k}\,x^k$$
$$-\,2x\sum_{k=0}^{n-1}(-1)^k\binom{2n-1}{2k+1}(1-x)^{n-1-k}\,x^k$$

By some calculation as above, we have

$$2\sum_{k=0}^{n+1}(-1)^k\binom{2n+3}{2k+1}(1-x)^{n+1-k}\,x^k$$
$$=\sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k+1}(1-x)^{n-k}\,x^k+\sum_{k=0}^{n+2}(-1)^k\binom{2n+5}{2k+1}(1-x)^{n+2-k}\,x^k$$
$$+\sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k}(1-x)^{n-k}\,x^k-\sum_{k=0}^{n+2}(-1)^k\binom{2n+5}{2k}(1-x)^{n+2-k}\,x^k$$

We can obtain that

$$\sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k}(1-x)^{n-k}\,x^k-\sum_{k=0}^{n+2}(-1)^k\binom{2n+5}{2k}(1-x)^{n+2-k}\,x^k$$
$$=4x\sum_{k=0}^{n+1}(-1)^k\binom{2n+3}{2k+1}(1-x)^{n+1-k}\,x^k$$

And thus we have

$$(2-4x)\sum_{k=0}^{n+1}(-1)^k\binom{2n+3}{2k+1}(1-x)^{n+1-k}\,x^k+\sum_{k=0}^{n}(-1)^k\binom{2n+1}{2k+1}(1-x)^{n-k}\,x^k$$
$$=\sum_{k=0}^{n+2}(-1)^k\binom{2n+5}{2k+1}(1-x)^{n+2-k}\,x^k$$

Now, let

$$F_n(x) = \sum_{k=0}^{n} (-1)^k \binom{2n+1}{2k+1} (1-x)^{n-k} x^k$$

We have

$$F_{n+2}(x) = F_{n+1}(x)(2-4x) - F_n(x),\ F_0(x) = 1, F_1(x) = 3-4x.$$

As desired.

□

3 ACKNOWLEDGEMENTS

The author thanks the Research Council of the University of Garmsar for support.

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