Solving system of Volterra integro-differential equations via Collocation method based on the Genocchi polynomials

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ABSTRACT

We provide a numerical scheme to solve a system of Volterra integro-differential equations. This is achived using the approximation with the Genocchi polynomials and collocation method. Applying these techniques, the system of Volterra integro-differential equations to the algebraic equations system of the unknown Genocchi coefficients. Rewriting the approximated polynomials with the specified Genocchi coefficients, an accuracy numerical solutions will be obtained. Results are applied for two examples to illustrate the validity of the method.

KEYWORDS: Genocchi polynomials, System of Volterra integro-differential equations, Collocation method

1 INTRODUCTION

The theory and application of integral equations are important subjects within applied mathematics. Integral equations are used as mathematical models for solving many physical problems; This is motivations to solve this kind of equations. A class of integral equation is integro-differential equations that have been found to describe physical phenomena such as wind ripple in the desert, nonohydrodynamics, dropwise consideration, glass-forming process [1]. This paper presents the Genocchi polynomials method for solving the following system of Volterra integro-differential equations:

$$\sum_{j=1}^{M} P_{i,j}(x)\mathcal{D}^{n_{i,j}}y_j(x) = W_i(x) + \int_0^x \sum_{j=1}^{k} \mathcal{K}_{i,j}(x,t)y_j(t)dt,\ i=1,...,k \tag{1}$$

where among the M functions y_j , k unknown functions must be determined (some y_j may share the same function for certain j), $K_{ij}$are the integral kernel D_{ij}^{n} is the n_{ij}-th order derivative, $P_{ij}(x)$, $W_i(x)$ are the coefficient functions, respectively.

In the following, the required definitions are provided in Section 2. In Section 3, the Genocchi polynomials operational matrix of the integer order derivative is calculated, then the main method for solving the SVIDEs is stated, and finally the numerical examples are given to confirm the results.

2 PRELIMINARIES

Here, the definition and properties of Genocchi polynomials are presented.

Definition 1: [2,3] The Genocchi polynomial $G_n(x)$ is defined by the generating function $Q(x,t)$:

$$\begin{aligned} Q(x,t) &= \frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} \\ G_n(x) &= \sum_{k=0}^{n}\binom{n}{k} g_{n-k}x^k = 2B_n(x) - 2^{n+1}B_n(x) \end{aligned} \tag{2}$$

where $g_k = 2B_k - 2^{k+1}B_k$ is the Genocchi number and $B_n(x)$ is the Bernoulli polynomials.

Some of the important properties of Genocchi polynomials are as follows:

$$\frac{dG_n(x)}{dx} = nG_{n-1}(x),\ n \geq 1 \tag{3}$$

$$\frac{d^k G_n(x)}{dx^k} = \begin{cases} 0 & \text{for } n \leq k \\ k!\binom{n}{k} G_{n-k}(x) & \text{for } n > k, \end{cases} \qquad k, n \in \mathbb{N} \cup \{0\} \tag{4}$$

$$G_n(1) + G_n(0) = 0,\ n > 1 \tag{5}$$

3 MAIN RESULTS

In this section, using the above definitions, the Genocchi polynomials operational matrix of the integer order derivative is calculated and then, a method for solving the system of Volterra integro-differential equations by the collocation method is explained.

3.1 GENOCCGI POLYNOMIALS OPERATIONAL MATRIX OF INTEGER ORDER DIFFERENTIATION

We assume that the N× N matrix P_G^n is the Genocchi polynomials operational matrix of an n-th order derivative:

$$D^n G(x) = P_G^n G(x)$$

$$D^n \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_N \end{bmatrix} = \begin{bmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1N} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2N} \\ & \vdots & \vdots & \\ \mu_{N1} & \mu_{N2} & \cdots & \mu_{NN} \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_N \end{bmatrix} \tag{6}$$

where the Genocchi polynomials operational matrix can be obtained by the following theorem[4].
Theorem 1: Given a set of $G_i(x)$, $i = 1,...,N$, where $G_i(x)$, is Genocchi polynomial of i-th order. Then, the Genocchi polynomials operational matrix of integer order n-th derivative over the interval $[0,1]$ is the $N \times N$ matrix given by:

$$P_G^n G(x) = \begin{bmatrix} \mu_{11} & \mu_{12} & \cdots & \mu_{1N} \\ \mu_{21} & \mu_{22} & \cdots & \mu_{2N} \\ & \vdots & \vdots & \\ \mu_{N1} & \mu_{N2} & \cdots & \mu_{NN} \end{bmatrix}$$

$$\mu_{ij} = \begin{cases} 0 & \mathit{for}\ i < n+j-1 \\ \frac{1}{2(j!)}\left(\frac{i!g_{i-(j-1)-n}}{(i-(j-1)-n)!} + \sum_{r=j-1}^{i-n} \frac{i!g_{i-r-n}}{(i-r-n)!r!}\right) & \mathit{for}\ i \geq n+j-1 \\ & i,j \in \mathbb{Z}^+, n \in \mathbb{N} \cup \{0\} \end{cases} \tag{7}$$

3.2 COLLOCATION METHOD FOR SOLVING SYSTEM

We apply Genocchi polynomials operational matrix of integer order derivatives with collocation method to solve numerically the system of Volterra integro-differential equations (1). Since $P_{ij}(x)$, $W_i(x)$ are functions known a priori in the problem using collocation method, some terms are approximated with the corresponding approximation using N order of Genocchi polynomials,

$$\mathcal{D}_x^{n_{i,j}} y_j(x) \approx C_j^T P_G^{n_j} G(x)$$
$$\mathcal{K}_{i,j}(x,t) \approx G^T(x)\mathcal{K}_{i,j}G(t) \tag{8}$$

where $C_j^T, j = 1,\ldots,k,$ are the unknown Genocchi coefficients corresponding to the unknown functions $y_j(x)$. Therefore, main Volterra integro-differential equations (1) are transformed into the following set of algebraic equations:

$$\sum_{j=1}^{M} G^T(x) P_G^{n_{i,j}\,T} C_j = W_i(x) + \sum_{j=1}^{k} G^T(x)\mathcal{K}_{i,j} T^{(0,x)} C_j$$

where

$$T^{(0,x)} = \int_0^x G(t)G^T(t)dt = [\gamma_{n,m}]$$
$$T^{(0,1)} = \int_0^1 G(t)G^T(t)dt = [\gamma_{n,m}].\gamma_{n,m} \tag{9}$$
$$\gamma_{n,m} = \sum_{r=0}^{n-1} \frac{n_{(r)}}{(m+1)^{(r+1)}}(G_{n-r}(x)G_{m+1+r}(x) - G_{n-r}(0)g_{m+1+r})$$

Now, using collocation points with uniform mesh in the interval [0,1], i.e. $\upsilon = \frac{r}{n}$, $r = 0,\ldots,N,$ decreased to solving a system of $k \times N$ algebraic equations. Together with the given $k \times r$ initial conditions (if any):

$$y_j^q(a_p) = G^T(x)C_j = b_p; \qquad j = 1,\ldots,k; \qquad p = 1,\ldots,r; \qquad q = 0,\ldots,r \tag{10}$$

the Genocchi coefficients C_j^T can be solved and finally, the approximate solutions of the k unknown functions $y_j(x)$ will be obtained:

$$y_{j,N}^*(x) = C_j^T G(x) \approx y_j(x) \tag{11}$$

3.3 NUMERICAL EXAMPLES

Here, several examples of a system of linear Volterra integro-differential equations are solved using the proposed method for calculation Genocchi polynomials. Also, the maximum error function of the approximate function computed for an arbitrary N-order approximation which is adopted from [5]

$$e_\infty(N) = \|f(x) - f_N(x)\|_\infty = Max\|f(x) - f_N(x)\|,\ a \le x \le b \tag{12}$$

where $f(x)$ is the exact solution and f_N (x) is the approximate solution.

Example 1: Let the following system of linear Volterra integral equations [6]

$$y_1(x) = x + \frac{1}{2}x^3 + \frac{1}{2}x^4 - \frac{1}{5}x^5 + \int_0^x (t^2 - x)(y_1(t) - y_2(t))dt$$
$$y_2(x) = x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 + \int_0^x t(y_1(t) + y_2(t))dt \tag{13}$$

with $k = M = 2$. Using $N = 3$ Genocchi polynomials to approximate the solution, i.e:

$$y_i(x) = \sum_{n=1}^{3} C_{i,n} G_n(x), \; i = 1, 2. \tag{14}$$

with the proposed Genocchi polynomials method, (16) is transformed into a matrix equation:

$$\begin{aligned} &\Big(G^T(x)P_G^{0T} - \mathcal{K}_{11}T^{(0,x)}\Big)C_1 - \mathcal{K}_{12}T^{(0,x)}C_2 - W_1(x) = 0 \\ &\Big(G^T(x)P_G^{0T} - \mathcal{K}_{22}T^{(0,x)}\Big)C_2 - \mathcal{K}_{21}T^{(0,x)}C_1 - W_2(x) = 0. \end{aligned} \tag{15}$$

By solving this matrix equation with collocation method at the following set of collocation points: $x_0 = 0, x_1 = 1/3, x_2 = 2/3, x_3 = 1$, the Genocchi coefficients are obtained as follows:

$$C_1 = [C_{1,1}, C_{1,2}, C_{1,3}]^T = [\frac{1}{2}, \frac{1}{2}, 0]^T, \; C_2 = [C_{2,1}, C_{2,2}, C_{2,3}]^T = [\frac{1}{2}, \frac{1}{2}, \frac{1}{3}]^T$$

so, the exact solutions are:

$$\begin{aligned} y_1(x) &= C_{1,1}G_1(x) + C_{2,1}G_2(x) + C_{3,1}G_3(x) = \frac{1}{2}(1) + \frac{1}{2}(2x-1) + 0(3x^2 - 3x) = x \\ y_2(x) &= C_{2,1}G_1(x) + C_{2,2}G_2(x) + C_{3,2}G_3(x) = \frac{1}{2}(1) + \frac{1}{2}(2x-1) + \frac{1}{3}(3x^2 - 3x) = x^2 \end{aligned} \tag{16}$$

Example 2: Consider the following system of $k = M = 2$ linear Volterra integral equations [6]:

$$\begin{aligned} y_1(x) &= e^{2x}\left(-\frac{1}{2}x^2 + \frac{1}{4}x + 1\right) - \frac{3}{4}x - \frac{1}{4} + \int_0^x xty_1(t)dt + \int_0^x (x+t)y_2(t)dt \\ y_2(x) &= e^{-2x}\left(2x^2 + x + \frac{5}{4}\right) - \frac{1}{4}e^{2x} - \frac{1}{2}x^2 + \int_0^x (x-t)y_1(t)dt + \int_0^x (x+t)^2 y_2(t)dt \end{aligned} \tag{17}$$

with exact solutions $y_1(x) = e^{2x}$ and $y_2(x) = e^{-2x}$.
Figure 1,2 show exact and approximate solutions for $N = 4,8$.

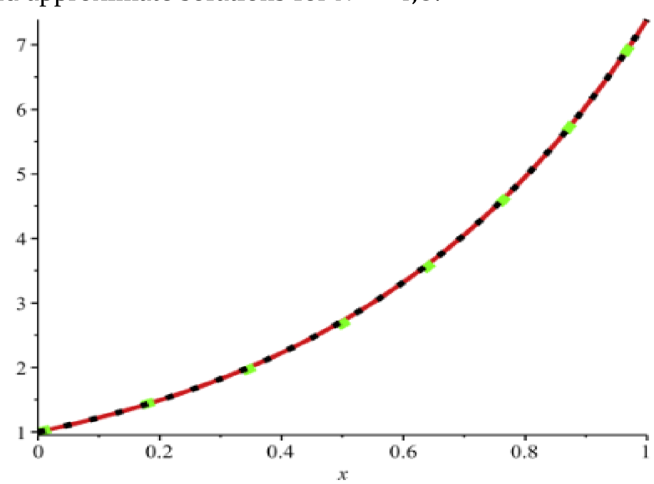

Figure1: Graph of Exact ( ), Approximate( ) $y_1(x)$ for $N = 4$ and Approximate( ) for $N = 8$.

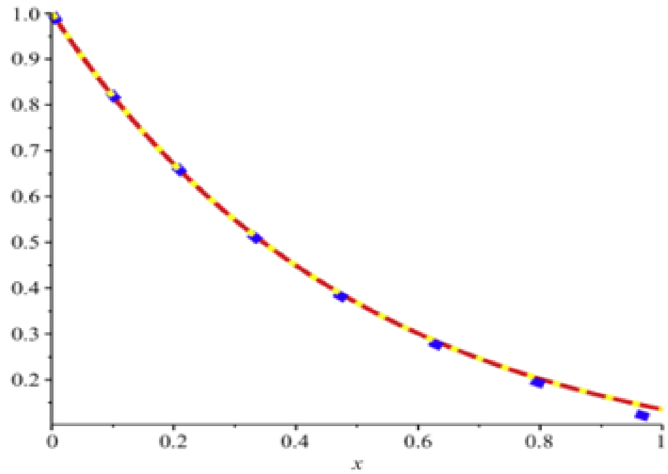

Figure1: Graph of Exact ( ), Approximate( ) $y_2(x)$ for $N = 4$ and Approximate( ) for $N = 8$.

Table 1 and 2 show the exact solutions and maximum error functions that compared to the results of [6], according to Euler polynomials, we see that better results are obtained with less processing time.

Table 1: Approximate y_1 for $N = 8$

x	Exact y_1	Genocchi y^*_1	Error y_1
0.0	1.0000000000000000	0.9999999999999800	1.9984 E14
0.2	1.4918246976412700	1.4918246986504000	1.0087 E09
0.4	2.2255409284924600	2.2255409519419400	2.3449 E08
0.6	3.3201169227365400	3.3201169271910500	4.4545 E09
0.8	4.9530324243951100	4.9530325263124700	1.0192 E07
1	7.3890560989306500	7.3890563758300900	2.7690 E07

Table 2: Approximate y_2 for $N = 8$

x	Exact y_2	Genocchi y^*_2	Error y_2
0.0	1.0000000000000000	0.9999999999999800	1.60000 E14
0.2	0.6703200460356390	0.6703200530048440	6.96920 E09
0.4	0.4493289641172210	0.4493289941169280	2.99997 E08
0.6	0.3011942119122020	0.3011942586124300	4.67002 E08
0.8	0.2018965179946550	0.2018966504944940	1.32499 E07
1	0.1353352832366120	0.1353356434809080	3.60244 E07

4 CONCLUSION

We use Genocchi polynomials operational matrix of integer order derivative with collocation method to solve a system of Volterra integro-differential equation. The comparison of the results with the Euler polynomial shows that the proposed Genocchi polynomials operational matrix method able to achieve the same degree or larger accuracy by using a comparatively smaller number N of polynomials bases. Furthermore, abled to solve the problem by using lesser CPU time.

REFERENCES

[1] P.K. Sahu, S. Saha Ray, "Legendre wavelets operational method for the numerical solutions of nonlinear Volterra integro-differential equations system" Appl. Math. Comput., 256 (2015), 715-723.

[2] B.M. Kim, L.C. Jang, W. Kim, H.I. Kwon, "Degenerate Changhee-Genocchi numbers and polynomials", Journal of Inequalities and Applications., 1 (2017), 1-10.

[3] D.S. Lee, H.K. Kim, "On the new type of degenerate poly-Genocchi numbers and polynomials", Advances in Difference Equations., 1 (2020), 1-15.

[4] A. Isah, C. Phang, P. Phang, "Collocation method based on Genocchi operational matrix for solving generalized fractional pantograph equations", International Journal of Differential Equations., (2017).

[5] A. Bhrawy, E. Tohidi And F. Soleymani, "A new Bernoulli matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals", Appl. Math. Comput., 219 (2012), 482–497.

[6] F. Mirzaee, S. Bimesl, "A new Euler matrix method for solving systems of linear Volterra integral equations with variable coefficients", J. Egypt. Math. Soc., 22 (2014), 238–248.