The second Chebyshev wavelets method for solving the nonlinear fractional Volterra-Fredholm integral-differential equations with a weakly singular kernel

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ABSTRACT

In this paper, a numerical method for solving of nonlinear fractional Volterra-Fredholm integral-differential equations with a weakly singular kernel is proposed. This method is based on the Second Chebyshev wavelets defined over [0, 1] combined with its operational matrices of fractional integration. The second Chebyshev wavelet operational matrix of fractional integration is derived and used to transform the nonlinear fractional Volterra-Fredholm integral-differential equations with a weakly singular kernel to a system of algebraic equations. Finally, some numerical example is shown to illustrate the accuracy and efficiency of the approach.

KEYWORDS: fractional Volterra-Fredholm integral-differential equation, weakly singular kernel, second Chebyshev wavelet, operational matrix.

1 INTRODUCTION

In this paper, we solve a nonlinear fractional Volterra-Fredholm integral-differential equations with a weakly singular kernel in the following form:

$$D^{\alpha}y_1(t)=\lambda_1\int_0^t \frac{[y(s)]^p}{(t-s)^{\beta}}ds+\lambda_2\int_0^1 k_1(t.s)[y(s)]^q ds+f_1(t) \quad y(0)=0 \tag{1}$$

where $y(t)$ is unknown function, functions $f(t)$ and $k(t,s)$ are known and λ_1,λ_2 are real constants and $p,q\in\mathbb{N}$. Here $0<\alpha,\beta<1$ and D^{α} denotes the Caputo fractional derivative [2].

2 FRACTIONAL CALCULUS

In this section, we review a short introduction of fractional calculus which will be used in this paper [1]. The Riemann-Liouville fractional integral operator is given by

$$I^{\alpha}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)d\tau. \qquad \alpha>0.$$

Where $\alpha\in[m-1.m].m\in\mathbb{N}$. The properties of this operator are as follows:

$$I_a^{\alpha}I_a^{\beta}f=I_a^{\alpha+\beta}f, \qquad I_a^{\alpha}I_a^{\beta}f=I_a^{\beta}I_a^{\alpha}f, \qquad I_a^{\alpha}t^c=\frac{\Gamma(c+1)}{\Gamma(c+\alpha+1)}t^{c+\alpha}.$$

The Caputo fractional derivative operator is given by

$$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau \qquad \alpha\in[n-1.n],$$

where $\in\mathbb{N}$. The properties between operator the Riemann-Liouville fractional integral operator and the Caputo fractional derivative is given by the following expression:

$$D^{\alpha} I^{\alpha} f(t) = f(t), \qquad I^{\alpha} D^{\alpha} f(t) = f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} t^k. \tag{2}$$

3 BLOCK PULSE FUNCTION

In this section, we define Block pulse function and their properties. The set of Block pulse function on [0,1) is defined as

$$b_i(t) = \begin{cases} 1 & \frac{i-1}{m} \le t < \frac{i}{m} \\ 0 & \text{otherwise.} \end{cases}$$

Where $i = 0,1,\dots,m-1$. Also, The vector Block pulse Function is obtained as follows:

$$B_m(t) = [b_1(t), b_2(t), \dots, b_m(t)]^T$$

and the important properties of the function is as follows :

$$b_i(t) b_j(t) = \begin{cases} b_i(t) & i = j \\ 0 & i \ne j, \end{cases} \qquad \int_0^1 b_i(t) b_j(t) dt = \begin{cases} \frac{1}{m} & i = j \\ 0 & i \ne j. \end{cases}$$

The Block pulse Function operational matrix of fractional integration F^{α} is obtation by:

$$I^{\alpha}\big(B_m(t)\big) \approx F^{\alpha} B_m(t),$$

with

$$F^{\alpha} = \frac{1}{m^{\alpha}} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \xi_2 & \cdots & \xi_{m-2} \\ 0 & 0 & 1 & \xi_1 & \cdots & \xi_{m-3} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

Where $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1},\ k = 1,2,\dots,m.$

A function $f \in L^2([0,1])$ can be approximate in terms of BPFs as the from

$$f(t) \approx \sum_{i=1}^{m'} f_i b_{m'}(t) = F^T B_{m'}(t).$$

Let $F = (f_{ij})$ and $G = (g_{ij})$ be two matrices of $m \times m$, so $F \otimes G$ is defined as $(F_{ij} \times G_{ij})_{m \times m}$. Suppose that $f(t), g(t) \in L_2[0,1)$, which can be expressed as $f(t) = F^T B_m(t)$ and $g(t) = G^T B_m(t)$ respectively, then $f(x), g(x)$ is obtained as follows:

$$f(x) g(x) = F^T B_m(x) B_m^T(x) G = H B_m(x)$$

where $H = F^T \otimes G^T$ [2].

4 THE SECOND CHEBYSHEV WAVELETS

In this section, we use the second kind Chebyshev polynomial to construct the second kind Chebyshev wavelet and give some properties of this wavelet. The second Chebyshev wavelets are defined on the interval [0,1) as:

$$\psi_{nm}(t) = \begin{cases} 2^{\frac{k}{2}} \sqrt{\frac{2}{\pi}} U_m\big(2^k t - 2n + 1\big) & \frac{n-1}{2^{k-1}} \le t < \frac{n}{2^{k-1}} \\ 0 & \text{otherwise,} \end{cases}$$

where $n = 1,2, \dots, 2^{k-1}, m = 0,1, \dots, M-1, k$ and M are positive integers and coefficient $\sqrt{\frac{2}{\pi}}$ is used for orthonormalitly. The function $U_m(t)$ is the second Chebyshev polynomial of degree m. Note that, these polynomials are defined on the interval $[-1,1]$ by the recurrence

$$U_0(t) = 1 \qquad U_1(t) = 2t \qquad U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t)$$

for $m = 1,2, \dots, M$. A function $f \in L^2([0.1])$ can be approximate in terms of the SCWs as

$$f(t) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\Psi(t) = \hat{f}(t)$$

where

$$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \dots, \psi_{1(M-1)}(t), \psi_{20}(t), \dots, \psi_{2(M-1)}(t), \dots, \psi_{2^{k-1}0}(t), \dots, \psi_{2^{k-1}(M-1)}(t)]^T$$

$$C = [c_{10}, c_{11}, \dots, c_{1(M-1)}, c_{20,} \dots, c_{2(M-1)}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}(M-1)}]^T.$$

We define the SCWs matrix$\Phi_{m\prime \times m\prime}$ as

$$\Phi_{m\prime \times m\prime} = [\Psi\left(\frac{1}{2m'}\right), \Psi\left(\frac{3}{2m'}\right), \dots, \Psi(\frac{2m\prime-1}{2m\prime})].$$

where $m' = 2^{k-1}M$. Also, there is a relation between BPFs and SCWs, namely

$$\Psi(t) = \Phi_{m\prime \times m\prime}B_{m\prime}(t). \tag{3}$$

Let

$$I^\alpha\Psi(t) \approx P^\alpha{}_{m'\times m'}\Psi(t), \qquad P^\alpha{}_{m'\times m'} = \Phi F^\alpha \Phi^{-1}. \tag{4}$$

where I^α is the Riemann-Liouville fractional integral operator of order α. The matrix $P^\alpha_{m\prime \times m\prime}$ is called the Chebyshev wavelets operational matrix of fractional integration [1].

5 METHOD ANALYSIS

For solving this equations, first we approximate $D^\alpha y(t)$, $f(t)$ and $k(t,s)$ in terms of SCW as follows

$$\begin{gathered} D^\alpha y(t) \simeq C^T\Psi(t) = C^T\Phi B_{m'}(t), \\ f(t) \simeq F^T\Psi(t) = F^T\Phi B_{m'}(t), \\ k(t,s) \simeq \Psi^T(t)K\Psi(s). \end{gathered} \tag{5}$$

From Eqs. (2), (3), (4) and (5), we obtain

$$y(t) = I^\alpha D^\alpha y(t) \simeq C^T P^\alpha \Psi(t), \tag{6}$$

$$[y(t)]^p = [C^T P^\alpha \Phi B_{m'}(t)]^p = E^P[B_{m'}(t)]^p = E^P B_{m'}(t). \tag{7}$$

Then there is

$$\int_0^t \frac{[y(s)]^p}{(t-s)^\beta} ds = E^p \int_0^t \frac{B_{m\prime}(s)}{(t-s)^\beta} ds = \Gamma(1-\beta)E^p I^{1-\beta} B_{m'}(t) = \Gamma(1-\beta)E^p F^{1-\beta} B_{m'}(t). \tag{8}$$

From Eq. (5) and $\int_0^1 B_{m\prime}(s)B_{m\prime}(s)^T ds = \frac{1}{m\prime}$, we have

$$\begin{aligned} \int_0^1 k(t,s)[y(s)]^q ds &= \int_0^1 \Psi^T(t)K\Psi(s)B_m(s)^T E^{q^T} ds \\ &= \Psi^T(t)K\Phi \int_0^1 B_{m\prime}(s)B_{m\prime}(s)^T ds E^{q^T} \\ &= \frac{1}{m'}\Psi^T(t)K\Phi E^{q^T} = \frac{1}{m'}E^q\Phi^T K^T \Phi B_{m'}(t). \end{aligned} \tag{9}$$

By substituting the Eqs. (5), (6), (8) and (9) into (1), we get

$$C^T\Phi B_{m'}(t) = \lambda_1\Gamma(1-\beta)E^p F^{1-\beta}B_{m'}(t) + \lambda_2 \frac{1}{m'}E^q\Phi^T K^T\Phi B_{m'}(t) + F^T\Phi B_{m'}(t). \tag{10}$$

Dispersing Eq. (28), we obtain

$$C^T\Phi = \lambda_1\Gamma(1-\beta)E^p F^{1-\beta} + \lambda_2 \frac{1}{m'}E^q\Phi^T K^T\Phi + F^T\Phi. \tag{11}$$

By solving system (11), we can get C_1. Then substituting them into (6), the unknown solutions can be obtained.

6 NUMERICAL EXAMPLE

To demonstrate the efficiency of this method, we consider the following a numerical example. Consider the fractional Volterra-Fredholm integral-differential equations with a weakly singular kernel

$$D^{\frac{1}{2}}y(t) = \int_0^t \frac{[y(s)]^2}{(t-s)^{\frac{1}{2}}} ds + \int_0^1 ts[y(t)]^2 ds + f(t) \quad y(0) = 0,$$

where $f(t) = -\frac{8}{3\Gamma(7)} t^{1.5} - 0.812698 t^{4.5} - \frac{t}{6}$. The exact solutions of the problem are $y(t) = t^2$. The absolute errors for $y(t)$ is listed Table 1 shows the absolute errors for different values of t.

Table 1: Absolute error for $M = 2$ and $k = 4,5$ of $y(t)$

	SCW		LW	
t	$M = 2. k = 4$	$M = 2. k = 5$	$M = 2. k = 4$	$M = 2. k = 5$
0	0	0	0	0
0.125	$9.5523e - 04$	$7.1123e - 04$	$1.5523e - 03$	$1.1482e - 03$
0.250	$3.6903e - 03$	$2.2886e - 03$	$5.5745e - 03$	$3.6323e - 03$
0.375	$7.4084e - 03$	$4.4354e - 03$	$1.1688e - 02$	$7.3545e - 03$
0.500	$1.2315e - 02$	$7.2896e - 03$	$2.1126e - 02$	$1.3004e - 02$
0.625	$1.9073e - 02$	$1.1266e - 02$	$3.6277e - 02$	$2.1999e - 02$
0.750	$2.9004e - 02$	$1.7142e - 02$	$6.1322e - 02$	$3.6856e - 02$
0.875	$4.4889e - 02$	$2.6584e - 02$	$1.0381e - 01$	$6.2416e - 02$

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