Perfect 2- coloring of the cuboctahedral and chvatal of 4 regular graph

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Abstract

A coloring of the vertex set V of graph $G = (V, E)$ with m colors, is called perfect if all color are used, and for all i, j the number of neighbors colors j of any vertex V of color i is a constant a_{ij}. The matrix $A = (a_{ij})_{i,j \in 1,\ldots,m}$ is called the parameter matrix .In this paper we study the Perfect 2- coloring of the cuboctahedral and chvatal 4 regular graph.

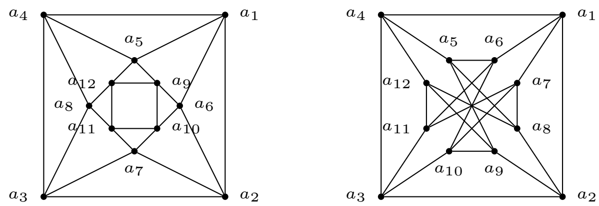
A cuboctahedral graph is the graph of vertices and edges of the cuboctahedron. It can also be constructed as the line graph of the cube.
The chvatal graph is an undirected graph with 12 vertices and 24 edges.
First, by applying the condition and limitation of the initial theorems, we identify the possible parameter matrices and then with the help of coloring our graphs, we examine their existence or non- existence.
Keywords: Connected graph, Parameter matrices, Perfect coloring, Equitable partition.

1 Introduction

Definition 1.1. *A vertex coloring of a graph is called perfect if each vertex of color i has exactly a_{ij} neighbors of color j.*

Definition 1.2. *For each graph G and each integer m, a mapping $T : V(G) \to 1, ..., m$ is called a perfect m- coloring with matrix $A = (a_{ij})_{i,j\in 1,...,m}$ if it is surjective, and for all i, j for every vertex of color i, the number of its neighbors of color j is equal to a_{ij}. The matrix A is called the parameter matrix of a perfect coloring. When $m = 2$, we denote the two colors by W and B representing white and black respectively.*

The cuboctahedral and chvatal graphs given as follow.



cuboctahedral graph chvatal graph

In this article, we enumerate parameter matrices perfect 2- coloring of the cuboctahedral and chvatal 4 regular graph. We present all possible parameter matrices for perfect 2- colorings 4 regular graph. In Section 2, we also prove which of matrices can have 2- perfect coloring in cuboctahedral graph. In Section 3, we prove 2- perfect coloring in chvatal graph.
First we have the following basic Theorems $1, ..., 5$ see [6] and then we obtain the parameter matrices for the perfect coloring of our graphs.

Theorem 1.3. *Suppose that G is a k- regular graph and T is a perfect m- coloring with matrix $A = (aij)$ in graph G. Then the sum of each row in matrix A is k.*

Theorem 1.4. *If T is a perfect coloring of the graph G with m colors, then any eigenvalue of the parameter matrix is an eigenvalue subset of the adjacent matrix G.*

Theorem 1.5. *Suppose that T is a perfect 2- coloring with $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in a connected graph G. Then a and b are opposite to zero.*

Theorem 1.6. *If W is the set of white vertex in a perfect 2- coloring of a graph G with matrix $A = (aij)i, j = 1, 2$, then $|W| = |V(G)|\frac{a_{21}}{a_{12}+a_{21}}$.*

Theorem 1.7. *Suppose that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is a parameter matrix of a perfect 2- coloring of a k- regular graph. Then eigenvalues of the parameter matrix are k and $a - c$ such that we obviously have $a - c \neq k$. So from Theorem 2.2 we conclude that $a - c$ is an eigenvalue of a k- regular connected graph which is not equal to k.*

By the given conditions, we can see that a parameter matrix of a perfect 2- coloring of cubic graphs may be one of the following matrices:

$$A_1 = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 4 \\ 2 & 2 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 4 \\ 3 & 1 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}, A_5 = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix}, A_6 = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}, A_7 = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix},$$

$$A_8 = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}, A_9 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, A_{10} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

And also in Theorem 1.4 we obtained a formula for calculating the number of white vertex in a perfect 2- coloring.

2 perfect 2- coloring of the cuboctahedral graph

In this section, with the help of the given theorems, we remove the parameter matrices that are not perfect coloring and examine the perfect 2- coloring of the cuboctahedral graph.

Theorem 2.1. *The cuboctahedral graph has a perfect* 2- *coloring with the matrix* $\begin{bmatrix} 0 & 4 \\ 2 & 2 \end{bmatrix}$.

Proof. By using the Theorem 1.4 the matrix A_2 can be a parameter matrix because it does have the condition of Theorem 1.4 with consideration eigenvalues adjacent matrix graph and using Theorems 1.2, 1.4.
The cuboctahedral graph has perfect 2- coloring with the matrix A_2. Using the Theorem 1.6 we have $|W| = 4$ and $|B| = 8$ so consider mapping T as follows:
$T(a_1) = T(a_3) = T(a_9) = T(a_{11}) = W$,
$T(a_2) = T(a_4) = T(a_5) = T(a_6) = T(a_7) = T(a_8) = T(a_{10}) = T(a_{12}) = B$.
It is clear that T is a perfect 2- coloring with the matrix A_2. □

Theorem 2.2. *There are no perfect* 2- *coloring of the cuboctahedral with the matrices* $A_5, A_7,$ *and* A_{10}.

Proof. A parameter matrix corresponding to perfect 2- coloring of the cuboctahedral may be one of the matrices A_5, A_7, and A_{10}. Using the Theorem 1.4 the matrix A_5 cannot be a parameter matrix, because of the Theorem 1.6 we have $|W| = 4$ which is a contradiction with the first row of A_7 we have $|W| = |B| = 6$ so using this number and considering entry the matrix A_7 , we being to color the vertex and examine different states.

Now have the following possibilities:
(1)$T(a_1) = T(a_2) = W$ and $T(a_6) = T(a_5) = T(a_4) = B$ then $T(a_{12}) = W$, which is a contradiction with the first row of A_7.
(2)$T(a_1) = T(a_5) = B$ and $T(a_2) = T(a_4) = T(a_6) = W$ so $T(a_9) = W$, which is a contradiction with the first row of A_7. Hence cuboctahedral has no perfect 2- coloring with the matrix A_7.
Also by using Theorems 1.6 for matrix A_{10} we have $|W| = |B| = 6$.
Now have the following possibilities:
(1)$T(a_1) = T(a_2) = T(a_5) = T(a_6) = W$ and $T(a_4) = B$ then $T(a_4) = B$, which is a contradiction with the second row of A_{10}.
(2)$T(a_1) = T(a_2) = T(a_5) = T(a_6) = B$ and $T(a_4) = W$ then $T(a_4) = W$, which is a contradiction with the first row of A_{10}. Hence cuboctahedral has no perfect 2- coloring with the matrix A_{10}.

□

Theorem 2.3. *The cuboctahedral graph has a perfect* 2- *coloring with the matrix* $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

Proof. By using the Theorem 1.4 the matrix A_9 can be a parameter matrix because it does have the condition of Theorem 1.4 with consideration eigenvalues adjacent matrix graph and using Theorems 1.2, 1.4.
The cuboctahedral graph has perfect 2- coloring with the matrix A_9. Using the Theorem 1.6 we have $|W| = 6$ and $|B| = 6$ so consider mapping T as follows:
$T(a_1) = T(a_2) = T(a_6) = T(a_8) = T(a_{11}) = T(a_{12}) = W$,
$T(a_3) = T(a_4) = T(a_5) = T(a_9) = T(a_{10}) = T(a_7) = B$.
It is clear that T is a perfect 2- coloring with the matrix A_9. □

3 Perfect 2- colorings the chvatal graph

In this section, with the help of the given theorems, we remove the parameter matrices that are not perfect coloring and examine the perfect 2- coloring of the cchvatal graph.

Theorem 3.1. *The chvatal graph has a perfect* 2- *coloring with the matrix* $\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$.

Proof. By using the Theorem 1.4 the matrix A_8 can be a parameter matrix because it does have the condition of Theorem 1.4 with consideration eigenvalues adjacent matrix graph and using Theorems 1.2, 1.4.
The chvatal graph has perfect 2- coloring with the matrix A_8. Using the Theorem 1.6 we

have $|W| = 4$ and $|B| = 8$ so consider mapping T as follows:
$T(a_1) = T(a_2) = T(a_3) = T(a_4) = W$,
$T(a_5) = T(a_6) = T(a_7) = T(a_8) = T(a_9) = T(a_{10}) = T(a_{11}) = T(a_{12}) = B$.
It is clear that T is a perfect 2- coloring with the matrix A_8. $\square$

Theorem 3.2. *There are no perfect* 2*- coloring of the chvatal with the matrices* A_1, A_3, A_5, A_6, *and* A_{10}.

Proof. A parameter matrix corresponding to perfect 2- coloring of the chvatal may be one of the matrices A_1, A_3, A_5, A_6, and A_{10}. Using the Theorem 1.4 the matrix A_1, A_3,and A_6 cannot be a parameter matrix, because of the Theorem 1.6 the number of white colors is not an integer. Also by using Theorems 1.6 for matrix A_5 we have $|W| = 3$, $|B| = 9$ but according to the entry matrix A_5, each W must be only adjacent to one W, and given the number of W and the shape of the graph, this is impossible.
Using Theorems 1.6 for matrix A_{10} we have $|W| = |B| = 6$. Now have the following possibilities:
(1)$T(a_1) = T(a_4) = T(a_6) = T(a_7) = W$ and $T(a_2) = B$ then $T(a_2) = B$, which is a contradiction with the second row of A_{10}.
(2)$T(a_1) = T(a_4) = T(a_6) = T(a_7) = B$ and $T(a_2) = W$ then $T(a_2) = W$, which is a contradiction with the first row of A_{10}. Hence chvatal has no perfect 2- coloring with the matrix A_{10}.

$\square$

Theorem 3.3. *The chvatal graph has a perfect* 2*- coloring with the matrix* $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$.

Proof. By using the Theorem 1.4 the matrix A_9 can be a parameter matrix because it does have the condition of Theorem 1.4 with consideration eigenvalues adjacent matrix graph and using Theorems 1.2, 1.4.
The chvatal graph has perfect 2- coloring with the matrix A_9. Using the Theorem 1.6 we have $|W| = 6$ and $|B| = 6$ so consider mapping T as follows:
$T(a_1) = T(a_2) = T(a_6) = T(a_9) = T(a_{11}) = T(a_{12}) = W$,
$T(a_3) = T(a_4) = T(a_5) = T(a_8) = T(a_{10}) = T(a_7) = B$.
It is clear that T is a perfect 2- coloring with the matrix A_9. $\square$

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