Lee Weight for Direct Sum of Codes

Farzaneh Farhang Baftani
Department of Mathematics, Ardabil Branch, **Islamic Azad University**, Ardabil, Iran.
Far_farhang2007@yahoo.com

ABSTRACT

Let C be a linear code of length n over $\mathbb{Z}_4$. The Lee support weight of C, denoted by $wt_L(C)$, is the sum of Lee weights of all columns of A(C) that A(C) is the $|C| \times n$ array of all code words in C. For $1 \leq r \leq rank(C)$, the r-th generalized Lee weight with respect to rank (GLWR) for C, denoted by d_r^L(C), is defined the minimum of all Lee weights of $\mathbb{Z}_4$-submodules of C with $rank = r$. In other words

$$d_r^L(\mathrm{C}) = min\{wt_L(D);\ \mathrm{D\ is\ a\ } \mathbb{Z}_4 - \mathrm{submodule\ of\ } C, \mathrm{rank(D)} =\ r\ \}.$$

For linear codes C_1 and C_2 over $\mathbb{Z}_4$ of length $n_1$and n_2, respectively, the Direct Sum of them ,denoted by $C_1 \oplus C_2$, is defined as follows:

$$C_1 \oplus C_2 = \{(c_1, c_2);\ c_1 \in C_1, c_2 \in C_2\}.$$

Motivated by finding $d_r^L(C_1 \oplus C_2)$ in terms of $d_r^L(C_1)$ and $d_r^L(C_2)$, we investigated $d_r^L(C_1 \oplus C_2)$ and we obtained $d_r^L(C_1 \oplus C_2)$ for $r = 1,2$. Moreover, we generally obtained an upper bound for $d_r^L(C_1 \oplus C_2)$ for all r, $1 \leq r \leq rank(C_1 \oplus C_2)$.

EYWORDS: Linear code, Hamming Weight, Lee Weight, Generalized Lee Weight, Direct Sum of Codes.

INTRODUCTION

Let $\mathbb{Z}_m$ be alphabet. The Lee Weight of an integer i, for $0 \leq i \leq m$ is defined as follows:

$$w_L(i) = min\{i, m - i\}.$$

The Lee metric on $\mathbb{Z}_m^n$ is defined by

$$w_L(a) = \sum_{i=1}^{n} w_L(a_i) \qquad (1)$$

Where the sum is defined in $\mathbb{N}_0$. We define Lee distance by

$$d_L(x, y) = w_L(x - y).$$

Note that in $\mathbb{Z}_4$, we have $w_L(0) = 0, w_L(1) = w_L(3) = 1, w_L(2) = 2$.

For more information, see [6]. The concept of Generalized Lee weight for codes over $\mathbb{Z}_4$, introduced by S. T. Dougherty in his seminal paper [1] for the first time. Then this concept was investigated by several authors, for example see [8]. This work is similar to what V. Wei did in [7] for Hamming weight and named it as Generalized Hamming Weight (GHW). This recent concept has been studied by several authors, see [2], [3], [4] and [5].

A code of length n over $\mathbb{Z}_4$ is a subset of the free module $\mathbb{Z}_4^n$ and the code is linear if it is a $\mathbb{Z}_4$– submodule of $\mathbb{Z}_4^n$.

Suppose that C is a code of length n over ring $\mathbb{Z}_4$. The rank of C which is denoted by $rank(C)$, is defined to be the minimum number of generators of C. For more information, see [1].

For $1 \leq r \leq rank(C)$, we define the r-th generalized Lee weight with respect to $rank$ (GLWR) for C, denoted by $d_r^L(C)$, as follows

$$d_r^L(C) = min\{wt_L(D): D \ is \ a \ \mathbb{Z}_4 - submodule \ of \ C \ with \ rank(D) = r\}.$$

Let C be a linear code of length n over $\mathbb{Z}_4$. Let $A(C)$ be the $|C| \times n$ array of all code words in C. Each column of A(C) corresponds to the following three cases:

i) The column contains only 0
ii) The column contains 0 and 2 equally often
iii) The column contains all elements of $\mathbb{Z}_4$ equally often,

we define the Lee support weight of these columns by 0, 2 and 1, respectively. Then we define Lee support weight ,$wt_L(C)$, by the sum of the Lee support weight of all columns of $A(C)$. For example, if

$$C = \{(2,0,0), (1,0,2), (0,0,0), (3,0,2), (3,2,2), (1,2,0), (0,2,2), (2,2,0)\}$$

So we have

$$A(C) = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \\ 3 & 0 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 0 \\ 0 & 2 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$

If c_i denotes the i-th column of C so we have $wt_L(c_1)$=1, $wt_L(c_2)$=2 and $wt_L(c_3)$=2 . Then we obtain $wt_L(C) = 1 + 2 + 2 = 5$. It is easy to show that if C is generated by one vector x, then $wt_L(C) = w_L(x)$. In this paper we will show by $C = [n, k]$, the code C of length n and $rank = k$.

1 MAIN RESULTS

Definition 1.1. Let C_i be an $[n_i, k_i]$linear code over $\mathbb{Z}_4$, for $i = 1,2$. Then the direct sum of C_1 and C_2 defined by

$$C_1 \oplus C_2 = \{(c_1, c_2);\ c_1 \in C_1, c_2 \in C_2\}$$

is an [$n_1 + n_2, k_1 + k_2$]-linear code over $\mathbb{Z}_4$.

Theorem 1.2.[1] Let C_1 and C_2 be an $[n;\ k_1, k_2]$ codes over $\mathbb{Z}_4$, then we have

$$w_L(C) = \frac{4}{|C|} \sum_{x \in C} (w_L(x) - wt(x)).$$

Theorem 1.3.(main result) Let C_1 and C_2 be linear codes over $\mathbb{Z}_4$, then we have

$$d_1^L(C_1 \oplus C_2) = \min\{d_1^L(C_1), d_1^L(C_2)\}.$$

Proof. Let $d_1^L(C_1) \leq d_1^L(C_2)$. We will show that $d_1^L(C_1 \oplus C_2) = d_1^L(C_1)$.

Suppose that $d_1^L(C_1) = w_L(c_1)$, where $c_1 \in C_1$. So, $(c_1, 0) \in C_1 \oplus C_2$. Since $< (c_1, 0) >$ is of rank one as a submodule of $C_1 \oplus C_2$ and its Lee weight satisfy

$$\{w_L(H); H \leq C_1 \oplus C_2, rank(H) = 1\},$$

also we have

$$\min\{w_L(H); H \leq C_1 \oplus C_2, rank(H) = 1\} = d_1^L(C_1 \oplus C_2),$$

so we obtain $d_1^L(C_1 \oplus C_2) \leq w_L(c_1, 0)$. We notice that by using equation (1), we have

$$w_L(c_1, 0) = w_L(c_1) + w_L(0) = w_L(c_1) = d_1^L(C_1) \Rightarrow$$

$$d_1^L(C_1 \oplus C_2) \leq d_1^L(C_1). \qquad (2)$$

Now let $d_1^L(C_1 \oplus C_2) = w_L(D)$, in which $rank(D) = 1$. Suppose that $D =< (c_1, c_2) >$ where $c_1 \in C_1$ and $c_2 \in C_2$. Therefore, by using equation (1) and definition of r-th GLWR, we have

$$w_L(D) = w_L\big((c_1, c_2)\big) = \begin{cases} w_L(c_2) \geq d_1^L(C_2) \geq d_1^L(C_1), & c_1 = 0 \\ w_L(c_1) \geq d_1^L(C_1), & c_2 = 0 \\ w_L(c_1) + w_L(c_2) \geq d_1^L(C_1) + d_1^L(C_2) \geq d_1^L(C_1), & c_1, c_2 \neq 0 \end{cases}$$

Hence we obtain

$$d_1^L(C_1 \oplus C_2) \geq d_1^L(C_1) \qquad (3)$$

By using equations (2) and (3), we have $d_1^L(C_1 \oplus C_2) = d_1^L(C_1)$. It is desired.

Theorem 1.4. Let C_1 and C_2 be linear codes over $\mathbb{Z}_4$, then we have

$$d_2^L(C_1 \oplus C_2) = \min\{d_2^L(C_1), d_2^L(C_2), \frac{1}{4}[d_1^L(C_1) + d_1^L(C_2)]\}.$$

Proof. Let $d_2^L(C_1) = w_L(D_1), D_1 =< x_1, y_1 >$ and $d_2^L(C_2) = w_L(D_2), D_2 =< x_2, y_2 >$. Let

$D =< (x_1, 0), (y_1, 0) >$. By using theorem (1.2), we have

$$w_L(D) = \frac{4}{|D|} \sum_{(\alpha,\beta)\in D} \big(w_L(\alpha,\beta) - wt(\alpha,\beta)\big) =$$
$$\tfrac{4}{|D_1|} \textstyle\sum_{\gamma\in D_1} (w_L(\gamma) - wt(\gamma)) = w_L(D_1) = d_2^L(C_1).$$

Note that $w_L(\alpha,\beta) = w_L(\alpha) + w_L(\beta), wt(\alpha,\beta) = wt(\alpha) + wt(\beta)$. Since D is of rank two and it is a submodule of $C_1 \oplus C_2$, satisfying

$$\{w_L(H); H \le C_1 \oplus C_2, rank(D) = 2\},$$

Also we have

$$\min\{w_L(H); H \le C_1 \oplus C_2, rank(D) = 2\} = d_2^L(C_1 \oplus C_2),$$

so we have

$$d_2^L(C_1 \oplus C_2) \le w_L(D)(= d_2^L(C_1)\,) \Rightarrow$$
$$d_2^L(C_1 \oplus C_2) \le d_2^L(C_1) \qquad (4).$$

By using the above method for $D' =< (0, x_2), (0, y_2) >$, we obtain

$$d_2^L(C_1 \oplus C_2) \le d_2^L(C_2) \quad (5)$$

Let $d_1^L(C_1) = w_L(x); x \in C_1$ and $d_1^L(C_2) = w_L(y); y \in C_2$. For $D =< (x, 0), (0, y) >$, we have

$$w_L(D) = \frac{4}{|D|} \sum_{(\alpha,\beta)\in D} \big(w_L(\alpha,\beta) - wt(\alpha,\beta)\big) = \frac{4}{|D|} \sum_{c_1,c_2\in Z_4} w_L(c_1x, c_2y) - wt(c_1x, c_2y)$$
$$= \frac{4}{|D|} \sum_{c_1,c_2\in\mathbb{Z}_4} w_L(c_1x) + w_L(c_2y) - wt(c_1x) - wt(c_2y) = \frac{1}{4}[d_1^L(C_1) + d_1^L(C_2)]$$

$$\Rightarrow d_2^L(C_1 \oplus C_2) \le \tfrac{1}{4}[d_1^L(C_1) + d_1^L(C_2)] \quad (6).$$

By equations (4), (5) and (6), we obtain

$$d_2^L(C_1 \oplus C_2) \le \min\{d_2^L(C_1), d_2^L(C_2), \tfrac{1}{4}[d_1^L(C_1) + d_1^L(C_2)]\} \qquad (7).$$

On the other hand, let $d_2^L(C_1 \oplus C_2) = w_L(D); D =< (\alpha_1, \beta_1), (\alpha_2, \beta_2) >$. Suppose that i number of $\{\alpha_1, \alpha_2\ \}$ and j number of $\{\beta_1, \beta_2\}$ are independent. This means that $j = 2 - t, t < i \le 2$. We have following three cases:

i) If $i + j > 2$ then we have

$$d_2^L(C_1 \oplus C_2) = w_L(D) = \frac{4}{|D|} \sum_{c_1,c_1 \in \mathbb{Z}_4} w_L(c_1\alpha_1 + c_2\alpha_2, c_1\beta_1 + c_2\beta_2) - wt(c_1\alpha_1 + c_2\alpha_2, c_1\beta_1 + c_2\beta_2)$$
$$> \frac{1}{4}[d_1^L(C_1) + d_1^L(C_2)] \geq d_2^L(C_1 \oplus C_2).$$

Hence $d_2^L(C_1 \oplus C_2) > d_2^L(C_1 \oplus C_2)$ which it is a contradiction.

ii) If $i + j < 2$, so we have $i \neq 0$ since D is linearly independent. Let $i = 1$ and $j = 0$. Hence $D =< (\alpha_1, 0), (k\alpha_1, 0) >$. Assume

$$r_1(\alpha_1, 0) + r_2(k\alpha_1, 0) = 0 \quad (8)$$

so we have

$$r_1\alpha_1 + r_2k\alpha_1 = 0$$

$$\Rightarrow r_1 + r_2k = 0$$

This means that equation (8) has non-zero solution, which it is a contradiction since D is linearly independent. Therefore we should have $i + j = 2$.

iii) If $i + j = 2$, let $i = t$ and $j = 2 - t$. Hence we have

$$d_2^L(C_1 \oplus C_2) = w_L(D) = \begin{cases} \frac{1}{4}[d_1^L(C_1) + d_1^L(C_2)], & i = j = 1 \\ d_2^L(C_2) & i = 0, j = 2 \\ d_2^L(C_1) & i = 2, j = 0. \end{cases}$$

Therefore, we obtain that

$$d_2^L(C_1 \oplus C_2) \geq \min\{d_2^L(C_1), d_2^L(C_2), \tfrac{1}{4}[d_1^L(C_1) + d_1^L(C_2)]\} \qquad (9).$$

Equations (7) and (9) complete the proof.

Theorem 2.5.(main result) Let C_i be linear codes of length n over $\mathbb{Z}_4$ for $i = 1,2$. Then we have

$$d_r^L(C_1 \oplus C_2) \leq \min\left\{\frac{1}{4^{r-t}} d_t^L(C_1) + \frac{1}{4^t} d_{r-t}^L(C_2), \qquad 0 \leq t \leq r\right\}.$$

Proof. Consider $D_1$and D_2 be submodules of rank t and $r - t$, respectively, where

$$d_t^L(C_1) = wt_L(D_1); D_1 =< x_1, x_2, \ldots, x_t >, x_i \in \mathbb{Z}_4 ,$$

$$d_{r-t}^L(C_2) = wt_L(D_2);\ D_2 =< y_1, y_2, \ldots, y_{r-t} >,\ y_i \in \mathbb{Z}_4.$$

Let $D =< (x_1, 0), \ldots ,\ (x_t, 0), (0, y_1), \ldots, (0, y_{r-t}) >$. So we have $rank(D) = r$. We have

$$wt_L D = \frac{4}{|D|}\left[\sum_{i=1}^{t} w_L(x_i) - wt(x_i) + \sum_{j=1}^{r-t} w_L(y_j) - wt(y_j)\right] =$$

$$\frac{1}{4^{r-t}} d_t^L(C_1) + \frac{1}{4^t} d_{r-t}^L(C_2).$$

Since D is of rank r, satisfying

$$\{w_L(H); H \leq C_1 \oplus C_2, rank(D) = r\},$$

also we have

$$\min\{w_L(H); H \leq C_1 \oplus C_2, rank(D) = r\} \,(= d_r^L(C_1 \oplus C_2)),$$

so we obtain $d_r^L(C_1 \oplus C_2) \leq\ wt_L D = \frac{1}{4^{r-t}} d_t^L(C_1) + \frac{1}{4^t} d_{r-t}^L(C_2)$. The proof is completed.

ACKNOWLEDGEMENTS

I'm thankful to all members of the conference committee who make great attempt to hold this conference every year. Thank you so much.

REFERENCES

[1] S.T. Dougherty, M. K. Gupta, K. Shiromoto, "On Generalized Weights for codes over $\mathbb{Z}_k$," Australian Journal of Combinatorics, Vol. 31, pp 231-248, 2005.

[2] S. T. Dougherty and S. Han, "Higher Weights and Generalized MDS Codes," Korean Math. Soc ,No. 6, pp 1167-1182, 2010.

[3] F. Farhang Baftani and H. R. Maimani, "The Weight Hierarchy of Hadamard Codes," Facta UniverSitatis (NIS)., .Vol34, No. 4, pp 797–803, 2019.

[4] F. Farhang Baftani and H. R. Maimani, "The weight hierarchy of Ham(r, q) ",Italian Journal of Pure and Applied Mathematics, N.45, pp 645–650..۲۰۲۱ ,

[5] G. L. Feig, K. K. Tzeng and V. K. Wei, "On the Generalized Hamming Weights of Several Classes of Cyclic Codes," IEEE Trans. Inform. Theory, Vol. 38, No. 3, pp 1125-1126, May, 1992.

[6] J. H. Van Lint, "Introduction to coding theory," Springer-Verlag, 1999.

[7] V.K.Wei, "Generalized Hamming Weights for linear codes," IEEE Trans. Inform. Theory Vol. 37, No.5, pp 1412-1418.۱۹۹۱ ,

[8] B.Yildiz, Z. Odemis Ozger, "A Generalization of the Lee Weight to $\mathbb{Z}_{p^k}$, TWMS J. App. Eng. Math. Vol. 2, No 2, pp145-153, 2012.