6th International Conference on
Combinatorics, Cryptography, Computer Science and Computing
November. 17-18, 2021

The 4-order variations on the 4-Fibonacci universal code

Mansour Hashemi, Elahe Mehraban

Department of Pure Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran
m_hashemi@guilan.ac.ir, e.mehraban.math@gmail.com

ABSTRACT

In this paper, first we study the 4-order variations on the 4-Fibonacci universal code and get tables $GH_a(4,n)$ for $-10 \le a \le -2$ and $1 \le n \le 40$. Also, we examine a new blocking algorithm using the 4-Fibonacci code and Gopala-Hemachandra code.

.

KEYWORDS: Gopala-Hemachandra, *4*-Fibonacci sequence.

1 INTRODUCTION

In mathematics, there are a lot of integer sequences, which are used in almost every field modern sciences. Many authors studied on these sequences (see [1,2]).

Definition 1.1. The 4-Fibonacci sequence, $\{F_n\}_0^\infty$ is defintded by:

$$F_n = F_{n-1} + F_{n-2} + F_{n-3} + F_{n-4}, \quad n \ge 4,$$

with the initial conditions $F_0 = F_1 = F_2 = 0$ and $F_3 = 1$ (see[3]). For example $\{F_n\}_0^\infty = \{0,0,0,1,1,2,4,8,\cdots\}$.

In 1986 , Apostolic and Fraenkle introduced the Fibonacci code (one of the type of the universal code used to encoding positive integers is the Fibonacci code) which used in the source coding as well as in cryptography[4]. In [5], Zeckendrof proved that every positive integer has a unique representation as the sum of the nonconsecutive element of Fibonacci numbers. Let A be a positive integer. Then, A can be written as a binary string $r_1 \cdots r_s$ with the length s such that $A = r_1 f_1 + r_2 f_2 + \cdots + r_s f_s$. In this method, for A, we first get the largest Fibonacci number f_s smaller or equal to A. Then, continue recursively with $A - f_s$. For example,

$$15=0\times 1+1\times 2+0\times 3+0\times 5+0\times 8+1\times 13,$$

so, 010001 is the Fibonacci representation for 15. In [6], studied on the 3-Fibonacci representation. Based on this fact, for k=3, we have $9 = 0\times\ 1 + 1\times 2 + 0\times 4 + 1\times 7.$ Therefore, 0101 is the 3-Fibonacci representation for 9.

The 4-Fibonacci code is gotten to add 3 bits to the 4-Fibonacci representation.

For example, we have $13 = 1\times 1 + 0\times 2 + 1\times 4 + 1\times 8$ for k=4. So, 1011 is the 4-Fibonacci representation and 1011111 is 4-Fibonacci code for 13.

In Tables 1, we obtain the 4-Fibonacci representation and the 4-Fibonacci code $1 \le n \le 40$, respectively.

Section 2, devotes to studying the Gopala-Hemachandra (GH) sequence and tables for the (GH) code for k=4 are displayed. In Section 3, we use the 4-Fibonacci code and the Gopala-Hemachandra code to give a new blocking algorithm.

1. Gopala-Hemachandra (GH) sequence and codes

The aim of this section is to get the 4-order variant 4-Fibonacci sequence and the 4-order variations on the 4-Fibonacci universal code

The Gopala-Hemachandra (GH) sequence defined as follows:

$$a, b, a+b, a+2b, 2a+3b, ...,$$

for any pair a , b and the Fibonacci numbers are obtained for the case a=1 and b=2 (see[7]).

A variation to the Fibonacci sequence is the more general GH sequences [8]. In [1], authors introduced the second order variant Fibonacci sequence and got tits Gopala-Hemachandra GH code. Also, they found that Gopala-Hemachandra (GH) code exists for $-20 \le a \le -2$ and $1 \le n \le 50$.The second order variant Fibonacci sequence (2-Fibonacci sequence), denoted by $VF_a(2,n)$, is defined as the (GH) sequence $\{a,\ b,\ a+b,\ a+2b,\ 2a+3b,\cdots\}$ where $b = 1-a$. $VF_a(2,n)$ is as

$$VF_a(2,n) = VF_a(2,n-1) + VF_a(2,n-2),\ n \ge\ 3,$$

where $VF_a(2,1) = a\ \ (a \in \mathbb{Z}),\ VF_a(2,2) = 1-a.$

For example, we have $\{-3,4,1,5,\cdots\}$ for $a = -3.$

In [9], Daykin proved that only the Fibonacci sequence gives a unique Zeckendorf's representation for all positive integers. But the variant Fibonacci sequences allow for multiple Zeckendorf's representation for the same positive integers. In [8], was showed that there is no Zeckendorf's representation for n=5, 12.

A. Nalli and C. Ozylimaz studied the third order variant 3-Fibonacci sequence and get GH code for $a = -2,\ \ldots,\ -20,$ and $1 \le n \le\ 100$ (see[6]) .

Here, we define the 4-order variant 4-Fibonacci sequence.

Definition 2.1. The 4-order variant 4-Fibonacci sequence, denoted by $VF_a(4,n)$, is defined as the GH sequence $\{a,\ b,\ a+b,\ 2a+2b,\ 4a+4b,\ldots\}$ where b=1-a. $VF_a(4,n)$ is as follows:

$$VF_a(4,n)=VF_a(4,n-1)+VF_a(4,n-2)+VF_a(4,n-3)+VF_a(4,n-4),\ \ n> k,$$

where $VF_a(4,1)=a\ \ (a\in \square\ ),\ VF_a(4,2)=1-a,\ VF_a(4,3)=1, VF_a(4,4)=2$.

For example (i) for a=-2, we have $\{-2,\ 3,\ 1,\ 2,\ 4,\ 10,\ 17,\ldots\}$.

(ii) Let a=-3, we have $VF_{-3}(4,\ n)=\{-3,\ 4,\ 1,\ 2,\ 4,\cdots\}$.

Now, we get a new universal source code by the 4-order variant 4-Fibonacci sequences. We obtain the 4-order variations on the 4-Fibonacci universal code, denoted by $GH_a(4,n)$ and are listed in the Tables 2 and 3 for k=4, $1\le n\le 40$. There exists GH code for a=-3.

2. Blocking method as an application of Gopala-Hemachandra (GH) codes

Here, we consider the use of Gopala-Hemachandra (GH) codes and introduce a blocking method using of these codes.

Now, we explain the blocking algorithm.

(1) Construct a matrix P. We get the message in a square matrix P of size 2m. Note that between the two words put zero and the size of the message matrix is even. If the size of the message matrix is not even, then the remaining elements of the matrix is set zero.

(2) Block the matrix P. We divide the matrix P into the submatrices E^i $\left(1\le\ i\le m^2\right)$, of the size $2\times\ 2$ from left to right. The number of the submatrices E^i is denoted by e.

(3) We choose n, a such that $n=\left[\dfrac{e}{2}\right]$ and

$$a=\begin{cases}-e, & if\quad 2\le e\le 10,\\ 4, & if\ \ e\ge 10.\end{cases}$$

(4)By considering n, we get Table 4 according to module 28.

(5) We get $T_i=e_{11}^i\oplus e_{22}^i\quad (\mathrm{mod}\,2)$ where $1\le i\le m^2$

($\oplus$ is the string summation of elements to the modulo 2 and defined by
the elements of two strings are summed one by one from left to right to the mode 2. If the length of one of them is less than other one, we will add zero elements to shorter string for having strings with equal lengths. For example, 1011101 $\oplus$ 01110=1100101 (mod 2)).

(6) We calculate $G_i=GH_a(k,n)\oplus\ e_{12}^i\ \ (\ mod\ \ 2)$.

(7) We get the coding matrix $C=\left[T_i,G_i,e_{21}^i,e_{22}^i\right],\quad 1\le i\le m^2$.

(8) End of algorithm.

Decoding blocking algorithm:

(1) We get $x_1^i=T_i\oplus e_{22}^i\quad (\mathrm{mod}\,2)$, and $x_1^i\to e_{11}^i$.

(2) We calculate $x_2^i = G_i \oplus GH_a(k,n) \quad (\ mod\ \ 2)$

and replacing x_2^i by e_{12}^i.

(3) Construct E^i.

(4) We get P.

(5) End of algorithm.

Example 3.1. Consider the message " SUMMER IS HoT". According the blocking algorithm, we have the matrix P as follows:

$$P = \begin{bmatrix} S & U & M & M \\ E & R & 0 & I \\ S & 0 & H & o \\ T & 0 & 0 & 0 \end{bmatrix}.$$

We divide P. Therefore, we have

$$E^1 = \begin{bmatrix} S & U \\ E & R \end{bmatrix}, \quad E^2 = \begin{bmatrix} M & M \\ 0 & I \end{bmatrix}, \quad E^3 = \begin{bmatrix} S & 0 \\ T & 0 \end{bmatrix}, \quad E^4 = \begin{bmatrix} H & o \\ 0 & 0 \end{bmatrix},$$

and n=2 and a=-2. Also, by Tables 2 and 4 , we obtain

$$E^1 \to \begin{bmatrix} F_{21} & F_{23} \\ F_7 & F_{20} \end{bmatrix} \to \begin{bmatrix} 01101111 & 00011111 \\ 1111111 & 10101111 \end{bmatrix},$$

$$E^2 \to \begin{bmatrix} F_{15} & F_{15} \\ F_{29} & F_{11} \end{bmatrix} \to \begin{bmatrix} 00001111 & 00001111 \\ 000001111 & 1101111 \end{bmatrix},$$

$$E^3 \to \begin{bmatrix} F_{21} & F_{29} \\ F_{22} & F_{29} \end{bmatrix} \to \begin{bmatrix} 01101111 & 000001111 \\ 11101111 & 000001111 \end{bmatrix},$$

$$E^4 \to \begin{bmatrix} F_{21} & F_{17} \\ F_{29} & F_{29} \end{bmatrix} \to \begin{bmatrix} 01101111 & 0100111 \\ 000001111 & 000001111 \end{bmatrix}.$$

By relation $T_i = e_{11}^i \oplus e_{22}^i \quad (\mathrm{mod}\, 2)$, we have

$$T_1 = e_{11}^{11} \oplus e_{22}^{1} = 01101111 \oplus 10101111 = 11000000 \pmod 2,$$
$$T_2 = 00001111 \oplus 1101111 = 11010001,$$
$$T_3 = 01101111 \oplus 000001111 = 011010001,$$
$$T_4 = 01101111 \oplus 000001111 = 011010001.$$

Using the relation $G_i = GH_a(k,n) \oplus e_{12}^{i} \; (mod \; 2)$, we have

$$G_1 = GH_{-2}(4,2) \oplus e_{12}^{1} = 0001111 \oplus 00011111 = 00000001 \; (mod \; 2),$$
$$G_2 = 0001111 \oplus 00001111 = 0001000 \quad (mod \; 2),$$
$$G_3 = 0001111 \oplus 000001111 = 000110011 \quad (mod \; 2),$$
$$G_4 = 0001111 \oplus 0100111 = 0101000 \quad (mod \; 2).$$

Therefore, we have

$$C = \begin{bmatrix} 11000000 & 00000001 & 1111111 & 10101111 \\ 11010001 & 0001000 & 000001111 & 1101111 \\ 011010001 & 000110011 & 11101111 & 000001111 \\ 011010001 & 0101000 & 000001111 & 000001111 \end{bmatrix}.$$

Now, we obtain decoding blocking algorithm. First, by the relation $x_1^i = T_i \oplus e_{22}^{i} \pmod 2$, we get

$$x_1^1 = 01101111, \quad x_1^2 = 00001111, \quad x_1^3 = 01101111, \quad x_1^4 = 01101111.$$

Similarly, by the relation $x_2^i = G_i \oplus GH_a(k,n) \quad (mod \; 2)$, we have

$$x_2^1 = 00011111, x_2^2 = 00001111, \quad x_2^3 = 000001111, x_2^4 = 0100111.$$

Therefore, according to above information and Table 2, we obtain $E^i \quad 1 \le i \le 4$

$$E^1 \to \begin{bmatrix} 01101111 & 00011111 \\ 1111111 & 10101111 \end{bmatrix} \to \begin{bmatrix} F_{21} & F_{23} \\ F_7 & F_{20} \end{bmatrix},$$
$$E^2 \to \begin{bmatrix} 00001111 & 00001111 \\ 000001111 & 1101111 \end{bmatrix} \to \begin{bmatrix} F_{15} & F_{15} \\ F_{29} & F_{11} \end{bmatrix},$$
$$E^3 \to \begin{bmatrix} 01101111 & 000001111 \\ 11101111 & 000001111 \end{bmatrix} \to \begin{bmatrix} F_{21} & F_{29} \\ F_{22} & F_{29} \end{bmatrix},$$
$$E^4 \to \begin{bmatrix} 01101111 & 0100111 \\ 000001111 & 000001111 \end{bmatrix} \to \begin{bmatrix} F_{21} & F_{17} \\ F_{29} & F_{29} \end{bmatrix}.$$

By Table 4, we have

$$E^1 = \begin{bmatrix} S & U \\ E & R \end{bmatrix}, \quad E^2 = \begin{bmatrix} M & M \\ 0 & I \end{bmatrix}, \quad E^3 = \begin{bmatrix} S & 0 \\ T & 0 \end{bmatrix}, \quad E^4 = \begin{bmatrix} H & o \\ 0 & 0 \end{bmatrix},$$

Consequently,

$$P = \begin{bmatrix} S & U & M & M \\ E & R & 0 & I \\ S & 0 & H & o \\ T & 0 & 0 & 0 \end{bmatrix}.$$

So, we get the message " SUMMER IS HoT".

3. Conclusion

In [1,6], the 2-order and 3-order variations on the *k*-Fibonacci ($k = 2;3$) universal code were studied for $n = 1, 2, \ldots, 100$ and $-20 \leq a \leq -2$, respectively. In this paper, we obtained the *4*-order variations on the *4*-Fibonacci universal code for $1 \leq n \leq 40$ and $-10 \leq a \leq -2$. We can use the results in the cryptographic.

REFERENCES

[1] M. Basu, B. Prased, Coding theory on the (*m; t*)- extension of the Fibonacci *p*-numbers,
Discrete Mathematics, Algorithms and Applications. 3 (2011) 259-267.

[2] M. Hashemi, E. Mehraban, Fibonacci length and the generalized order *k*-Pell sequences of the 2-generator *p*-groups of nilpotency class 2, J. Algebra. Appl. (In press).

[3] M. Hashemi, E. Mehraban, Some New Codes on the *k*-Fibonacci Sequence, Mathematical
Problems in Engineering, Volume 2021, Article ID 7660902.

[4] A. Apostolic, A. Fraenkle, Robust transmission of strings using Fibonacci represntations,
IEEE Trans. Inform. Theory. 33 (1987) 238–245.

[5] E. Zeckendrof, Representation des nombres naturels par une somme des nombres de Fibonacci ou de nombres de lucas, Bull. Soc. Roy . Sci. Liege. (1972) 41179–182.

[6] A. Nalli, C. Ozyilmaz, The third order variations on the Fibonacci universal code, J. Number. Theory. 149 (2015) 15–32.

[7] I.G. Pearce, Indian mathematics: Redressing the balance, http;//www. history. mcs. st-andrews. ac.uk/ history/projects/pearce/index.html, (2002).

[8] S. Kak, The golden mean and the physics of aesthetics, Foarm Magazine. 5 (2006) 73–81.

[9] D. E. Daykin, Representation of natural numbers as sums of generalized Fibonacci numbers, J. Lond. Math. Soc. 35 (1960) 143–160.

Table 1: The 4-Fibonacci representation and the 4-Fibonacci code

n	The 4-Fibonacci representation	The 4-Fibonacci code
1	1	1111
2	01	01111
3	11	11111
4	001	001111
5	101	101111
6	011	011111
7	111	111111
8	0001	0001111
9	1001	1001111
10	0101	0101111
11	1101	1101111
12	0011	0011111
13	1011	1011111
14	0111	0111111
15	00001	00001111
16	10001	10001111
17	01001	01001111
18	11001	11001111
19	00101	00101111
20	10101	10101111
21	01101	01101111
22	11101	11101111
23	00011	00011111
24	10011	10011111
25	01011	01011111
26	11011	11011111
27	00111	00111111
28	10111	10111111
29	000001	000001111
30	100001	100001111
31	010001	010001111
32	110001	110001111
33	001001	001001111
34	101001	101001111
35	011001	011001111
36	111001	111001111
37	000101	000101111
38	100101	100101111
39	010101	010101111
40	110101	110101111

Table 2: The 4-order GH codes for $1 \le n \le 40$ and a=-2,-4,-5,-6

n	$GH_{-2}(4,n)$	$GH_{-4}(4,n)$	$GH_{-5}(4,n)$	$GH_{-6}(4,n)$
1	001111	001111	001111	001111
2	0001111	0001111	0001111	0001111
3	01111	0011111	0011111	0011111
4	00001111	00001111	00001111	00001111
5	00101111	01111	00101111	00101111
6	00011111	00011111	01111	00011111
7	01001111	0101111	011111	01111
8	100001111	100001111	100001111	100001111
9	01011111	101001111	101001111	101001111
10	000001111	100101111	100101111	100101111
11	001001111	01011111	01101111	01001111
12	000101111	000001111	01011111	100011111
13	010001111	001001111	000001111	01011111
14	000011111	000101111	001001111	000001111
15	1000001111	001101111	000101111	001001111
16	1010001111	000011111	001101111	00010111
17	0000001111	010001111	000011111	001101111
18	0010001111	011001111	001011111	000011111
19	0001001111	0000001111	010001111	1000101111
20	0100001111	0010001111	0000001111	1010101111
21	0000101111	0001001111	0010001111	0000001111
22	0010101111	0011001111	0001001111	0010001111
23	0001101111	0000101111	0011001111	0001001111
24	0100101111	0100001111	0000101111	0011001111
25	0110101111	0110001111	0010101111	0000101111
26	0101101111	0101001111	0100001111	0010101111
27	0000011111	0111001111	0110001111	0001101111
28	0010011111	0100101111	0101001111	0100001111
29	0001011111	0110101111	0111001111	0110001111
30	0100011111	0101101111	0100101111	0101001111
31	10000001111	0000011111	0110101111	0111001111
32	10100001111	0010011111	0101101111	0100101111
33	00000001111	10000001111	0000011111	0110101111
34	00100001111	10100001111	10000001111	0101101111
35	00010001111	10010001111	10100001111	10000001111
36	01000001111	10110001111	10010001111	10100001111
37	00001001111	00000001111	0000111111	10010001111
38	00101001111	00100001111	10001001111	10110001111
39	00011001111	00010001111	00000001111	10001001111
40	01001001111	00110001111	00100001111	10101001111

Table 3: The 4-order GH codes for $1 \le n \le 40$ and a=-7,-8,-9,-10

n	$GH_{-7}(4,n)$	$GH_{-8}(4,n)$	$GH_{-9}(4,n)$	$GH_{-10}(4,n)$
1	001111	001111	001111	001111
2	0001111	0001111	0001111	0001111
3	0011111	0011111	001111	0011111
4	00001111	00001111	00001111	00001111
5	00101111	00101111	00101111	00101111
6	00011111	00011111	00011111	00011111
7	00111111	00111111	00111111	00111111
8	01111	100001111	100001111	100001111
9	011111	01111	101001111	101001111
10	0101111	011111	01111	100101111
11	0111111	0101111	011111	01111111
12	01001111	100011111	0101111	011111
13	01101111	01001111	0111111	0101111
14	01011111	01101111	01001111	0111111
15	000001111	01011111	01101111	1000001111
16	001001111	000001111	01011111	1010001111
17	000101111	001001111	000001111	1001001111
18	001101111	0001001111	001001111	000001111
19	000011111	0011001111	000101111	001001111
20	001011111	000011111	001101111	000101111
21	000111111	001011111	000011111	001101111
22	0000001111	000111111	001011111	000011111
23	0010001111	0000001111	000111111	001011111
24	0001001111	0010001111	0000001111	000111111
25	0011001111	0001001111	0010001111	0000001111
26	0000101111	0011001111	0001001111	0010001111
27	0010101111	001011111	0011001111	0001001111
28	0001101111	001011111	0000101111	0011001111
29	0011101111	0001101111	0010101111	0000101111
30	0100001111	0011101111	0001101111	0010101111
31	0110001111	1000011111	010011111	00010101111
32	0101001111	0100001111	011011111	0011101111
33	0111001111	0110001111	010111111	1000011111
34	0100101111	0101001111	0100001111	1010011111
35	0110101111	0111001111	0110001111	1001011111
36	10000001111	0100101111	0101001111	0100001111
37	10100001111	10000001111	0111001111	0110001111
38	10010001111	10100001111	10000001111	0101001111
39	10110001111	10010001111	10100001111	10000001111
40	10001001111	10110001111	10010001111	10100001111

Table 4: English Alphabet

A	B	C	D	E	F	G	H	I
F_{n+1}	F_{n+2}	F_{n+3}	F_{n+4}	F_{n+5}	F_{n+6}	F_{n+7}	F_{n+8}	F_{n+9}
J	K	L	M	N	O	P	Q	R
F_{n+10}	F_{n+11}	F_{n+12}	F_{n+13}	F_{n+14}	F_{n+15}	F_{n+16}	F_{n+17}	F_{n+18}
S	T	U	V	W	X	Y	Z	0
F_{n+19}	F_{n+20}	F_{n+21}	F_{n+22}	F_{n+23}	F_{n+24}	F_{n+25}	F_{n+26}	F_{n+27}