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On the tree-number of conjugacy class graphs of some metacyclic groups

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Abstract

For a finite group G with $V(G)$ as the set of all non-central conjugacy classes of it, the conjugacy class graph $\Gamma(G)$ is defined as: its vertex set is the set $V(G)$ and two distinct vertices a^G and b^G are connected with an edge if $(o(a), o(b)) > 1$. In this paper, we determine the tree-number of the conjugacy class graphs of metacyclic groups of order less than thirty.

Keywords: Metacyclic group, conjugacy class graph, tree-number

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1 Introduction

A graph Γ is a pair (V, E), where V is a set whose elements are called vertices and E is a set of paired vertices, whose elements are called edges. Suppose that Γ be a graph with vertex set $V = \{v_1, \ldots, v_n\}$ and edge set $E = \{e_1, \ldots, e_m\}$. The adjacency matrix of Γ denoted by A, is an $n \times n$ matrix whose entries a_{ij} are 1, when v_i and v_j are adjacent and 0 otherwise. The degree of a vertex v_i is denoted by $deg(v_i)$ and the degree matrix denoted by Δ is defined as $\Delta = diag(deg(v_1), deg(v_2), \ldots, deg(v_n))$, which is a diagonal matrix. Then, the Laplacian matrix of Γ is denoted by Q which satisfies $Q = \Delta - A$. We denote by $\mu_0, \mu_1, \ldots, \mu_n$ the eigenvalues of the Laplacian matrix of Γ. It is proved in [3] that at least one of these eigenvalues is zero. Without loss of generality, we assume that $\mu_0 = 0$. The characteristic polynomial of the Laplacian matrix Q is $\sigma(\Gamma, \mu) = det(\mu I - Q)$. The tree-number of Γ is the number of spanning trees of Γ and is denoted by $\kappa(\Gamma)$. For disconnected graphs Γ, $\kappa(\Gamma)$ is defined 0 (see [3]).
Let G be a finite non-abelian group and $V(G)$ be the set of all non-central conjugacy classes of G. A conjugacy class graph $\Gamma(G)$ according to the orders of representatives of conjugacy classes is defined in [6] as below: its vertex set is the set $V(G)$ and two distinct vertices a^G and b^G are connected with an edge if $(o(a), o(b)) > 1$. A metacyclic group is an extension of a cyclic group by a cyclic group. Equivalently, a metacyclic group is a group G having a cyclic normal subgroup N, such that the quotient $\frac{G}{N}$ is also cyclic. Clearly, every cyclic group is metacyclic. There are some known results about metacyclic groups which we point them briefly. One can find the proofs in [4] and [5]. The subgroups and quotient groups of metacyclic groups are metacyclic. The direct product or semidirect product of two cyclic groups is metacyclic. So, the dihedral groups and the semi-dihedral groups are metacyclic. The dicyclic groups are metacyclic. Every finite group of squarefree order is metacyclic. Recall that $K \rtimes H$ is a semidirect product of K and H with normal subgroup K and $K \rtimes_f H$ is the Frobenius group with kernel K and complement H. All further unexplained notations are standard. In this paper, we compute the tree-number of conjugacy class graphs of metacyclic groups of order less than thirty.

2 Examples and Preliminaries

In this section, we give some examples and preliminary results that will be used in the proof of our main results.

Proposition 2.1. *([2]) The multiplicity of 0 as an eigenvalue of Q is equal to the number of connected components of the graph.*

Proposition 2.2. *([1]) The Laplacian matrix of the complete graph K_n has eigenvalues 0 with multiplicity 1 and n with multiplicity $n-1$.*

Corollary 2.3. *(Corollary 6.5 of [3]) Let $0 \leq \mu_1 \leq \ldots \leq \mu_{n-1}$ be the Laplacian spectrum of a graph Γ with n vertices. Then $\kappa(\Gamma) = \frac{\mu_1\mu_2\cdots\mu_{n-1}}{n}$.*

In the following we present some examples of metacyclic groups and find their characteristic Laplacian polynomials and eigenvalues.

Example 2.4. Since $\mathbb{Z}_4$ is a cyclic normal subgroup of Q_8 such that $|\frac{Q_8}{\mathbb{Z}_4}| = 2$, we deduce that Q_8 is metacyclic. Also, we have $\sigma(\Gamma, \mu) = \mu(\mu-3)^2$.

Example 2.5. Since $\mathbb{Z}_8$ is a cyclic normal subgroup of Q_{16} such that $|\frac{Q_{16}}{\mathbb{Z}_8}| = 2$, we deduce that Q_{16} is metacyclic. Also, we have $\sigma(\Gamma, \mu) = \mu(\mu-5)^4$.

Example 2.6. Since $\mathbb{Z}_8$ is a cyclic normal subgroup of $M_4(2)$ such that $|\frac{M_4(2)}{\mathbb{Z}_8}| = 2$, we deduce that $M_4(2)$ is metacyclic. Also, we have $\sigma(\Gamma, \mu) = \mu(\mu-6)^5$.

Example 2.7. Since $\mathbb{Z}_6$ is a cyclic normal subgroup of $\mathbb{Z}_3 \times S_3$ such that $|\frac{\mathbb{Z}_3 \times S_3}{\mathbb{Z}_6}| = 3$, we deduce that $\mathbb{Z}_3 \times S_3$ is metacyclic. Also, we have $\sigma(\Gamma, \mu) = \mu(\mu-2)(\mu-5)^2(\mu-6)^2$.

Example 2.8. Since $\mathbb{Z}_{12}$ is a cyclic normal subgroup of $\mathbb{Z}_4 \times S_3$ such that $|\frac{\mathbb{Z}_4 \times S_3}{\mathbb{Z}_{12}}| = 2$, we deduce that $\mathbb{Z}_4 \times S_3$ is metacyclic. Also, we have $\sigma(\Gamma, \mu) = \mu(\mu-3)(\mu-7)^3(\mu-8)^3$.

Example 2.9. Since $\mathbb{Z}_{12}$ is a cyclic normal subgroup of $\mathbb{Z}_3 \times D_8$ such that $|\frac{\mathbb{Z}_3 \times D_8}{\mathbb{Z}_{12}}| = 2$, we deduce that $\mathbb{Z}_3 \times D_8$ is metacyclic. Also, we have $\sigma(\Gamma, \mu) = \mu(\mu-9)^8$.

Example 2.10. Since $\mathbb{Z}_{12}$ is a cyclic normal subgroup of $\mathbb{Z}_3 \times Q_8$ such that $|\frac{\mathbb{Z}_3 \times Q_8}{\mathbb{Z}_{12}}| = 2$, we deduce that $\mathbb{Z}_3 \times Q_8$ is metacyclic. Also, we have $\sigma(\Gamma, \mu) = \mu(\mu-9)^8$.

Example 2.11. Since $\mathbb{Z}_6$ is a cyclic normal subgroup of $\mathbb{Z}_2 \times Dic_3$ such that $\frac{\mathbb{Z}_2 \times Dic_3}{\mathbb{Z}_6} \cong \mathbb{Z}_4$, we deduce that $\mathbb{Z}_2 \times Dic_3$ is metacyclic. Also, we have $\sigma(\Gamma, \mu) = \mu(\mu-3)(\mu-7)^3(\mu-8)^3$.

Example 2.12. Since $\mathbb{Z}_9$ is a cyclic normal subgroup of $(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_3$ such that $|\frac{(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_3}{\mathbb{Z}_9}| = 2$, we deduce that $(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_3$ is metacyclic. Also, we have $\sigma(\Gamma, \mu) = \mu(\mu-8)^7$.

Example 2.13. Since the direct product and semidirect product of two cyclic groups, the dihedral groups, the semi-dihedral groups and the dicyclic groups are metacyclic, we deduce that S_3, D_8, D_{10}, D_{12}, Dic_3, D_{14}, D_{16}, SD_{16}, $\mathbb{Z}_4 \rtimes \mathbb{Z}_4$, D_{18}, D_{20}, $\mathbb{Z}_5 \rtimes_f \mathbb{Z}_4$, Dic_5, $\mathbb{Z}_7 \rtimes_f \mathbb{Z}_3$, D_{22}, D_{24}, Dic_6, $\mathbb{Z}_3 \rtimes \mathbb{Z}_8$, D_{26}, Dic_7 and D_{28} are metacyclic groups.

3 Main results

Theorem 3.1. *Let G be a non-abelian metacyclic group of order least than 30 and $\Phi = (|g_1^G|, |g_2^G|, \ldots, |g_n^G|)$, such that g_i^G are the conjugacy classes of G for $1 \leq i \leq n$. Then the values of tree-numbers of $\Gamma(G)$ is given in Table 1.*

Table 1: Tree number of metacyclic groups of order less than 30

G	Φ	Orders of representatives of conjugacy classes of G	$\kappa(\Gamma)$
S_3	$(1,3,2)$	$(1,2,3)$	0
Q_8	$(1,2,2,1,2)$	$(1,4,4,2,4)$	3
D_8	$(1,2,2,1,2)$	$(1,2,4,2,2)$	3
D_{10}	$(1,5,2,2)$	$(1,2,5,5)$	0
D_{12}	$(1,3,2,2,3,1)$	$(1,2,6,3,2,2)$	3
Dic_3	$(1,3,1,2,3,2)$	$(1,4,2,3,4,6)$	3
D_{14}	$(1,7,2,2,2)$	$(1,2,7,7,7)$	0
D_{16}	$(1,4,2,2,1,4,2)$	$(1,2,8,4,2,2,8)$	125
Q_{16}	$(1,4,2,2,1,4,2)$	$(1,4,8,4,2,4,8)$	125
SD_{16}	$(1,4,4,2,1,2,2)$	$(1,4,2,4,2,8,8)$	125
$M_4(2)$	$(1,2,2,1,1,2,2,2,1,2)$	$(1,8,2,4,2,8,8,4,4,8)$	1296
$\mathbb{Z}_4 \rtimes \mathbb{Z}_4$	$(1,2,2,1,1,2,2,2,1,2)$	$(1,4,4,2,2,4,4,4,2,4)$	1296
D_{18}	$(1,9,2,2,2,2)$	$(1,2,9,3,9,9)$	0
$\mathbb{Z}_3 \times S_3$	$(1,3,1,2,3,1,2,3,2)$	$(1,2,3,3,6,3,3,6,3)$	300
D_{20}	$(1,5,1,2,5,2,2,2)$	$(1,2,2,5,2,10,5,10)$	192
$\mathbb{Z}_5 \rtimes_f \mathbb{Z}_4$	$(1,5,5,4,5)$	$(1,4,2,5,4)$	0
Dic_5	$(1,5,1,2,5,2,2,2)$	$(1,4,2,5,4,10,5,10)$	192
$\mathbb{Z}_7 \rtimes_f \mathbb{Z}_3$	$(1,7,3,7,3)$	$(1,3,7,3,7)$	0
D_{22}	$(1,11,2,2,2,2,2)$	$(1,2,11,11,11,11,11)$	0
D_{24}	$(1,6,2,1,2,6,2,2,2)$	$(1,2,4,2,3,2,12,6,12)$	5292
Dic_6	$(1,6,2,1,2,6,2,2,2)$	$(1,4,4,2,3,4,12,6,12)$	5292
$\mathbb{Z}_3 \rtimes \mathbb{Z}_8$	$(1,3,1,1,2,3,3,1,2,2,3,2)$	$(1,8,4,2,3,8,8,4,12,6,8,12)$	65856
$\mathbb{Z}_4 \times S_3$	$(1,3,1,1,2,3,3,1,2,2,3,2)$	$(1,2,4,2,3,4,2,4,12,6,4,12)$	65856
$\mathbb{Z}_3 \times D_8$	$(1,2,2,1,1,2,2,2,1,1,2,2,2,1,2)$	$(1,2,2,3,2,4,6,6,3,6,12,6,6,6,12)$	4782969
$\mathbb{Z}_3 \times Q_8$	$(1,2,2,1,1,2,2,2,1,1,2,2,2,1,2)$	$(1,4,4,3,2,4,12,12,3,6,12,12,12,6,12)$	4782969
$\mathbb{Z}_2 \times Dic_3$	$(1,3,1,1,2,3,3,1,2,2,3,2)$	$(1,4,2,2,3,4,4,2,6,6,4,6)$	65856
D_{26}	$(1,13,2,2,2,2,2,2,)$	$(1,2,13,13,13,13,13,13)$	0
$(\mathbb{Z}_3)^2 \rtimes \mathbb{Z}_3$	$(1,3,3,1,3,3,3,1,3,3,3)$	$(1,3,3,3,3,3,3,3,3,3)$	262144
Dic_7	$(1,7,1,2,7,2,2,2,2,2)$	$(1,4,2,7,4,14,7,14,7,14)$	34560
D_{28}	$(1,7,1,2,7,2,2,2,2,2)$	$(1,2,2,7,2,14,7,14,7,14)$	34560

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