A Note on Jacobsthal-Lucas-Leonardo Sequence

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ABSTRACT

In this paper we introduce a particular number sequence, namely Jacobsthal-Lucas-Leonardo sequence. Some interesting identities like as Cassini's identity, Binet formula and summation formulas about this sequence are given in this paper. In addition, we represent some examples related to these identities and summation formulas. Also, a PYTHON code to generate the first n terms of Jacobsthal-Lucas-Leonardo sequence is given in this paper.

KEYWORDS: Jacobsthal-Lucas sequence, Leonardo numbers, Binet Formula, Summation formula.

1 INTRODUCTION

In matrix algebras, combinatory theories, computer science, engineering and social sciences we can see the fundamental roles of special number sequences like as Fibonacci sequence, Lucas sequence, Jacobsthal-Lucas sequence and other number sequences. Therefore, numerous authors have widely studied these sequences. They proposed different results and identities about these sequences and illustrated various applications of these sequences. For example the authors in [9] investigated the eigenvalues and determinant of special circulant matrix involving (k,h)-Jacobsthal sequence and (k,h)-Jacobsthal-like sequence.

In this paper we introduce a particular number sequence, namely Jacobsthal-Lucas-Leonardo sequence which is a combination of Jacobsthal- sequence, Lucas sequence and Leonardo numbers. Some interesting identities like as Cassini's identity, Binet formula and summation formulas about this sequence are given in this paper. In addition, some examples related to these identities and summation formulas are represented in this paper. Also, a PYTHON code to generate the first n terms of Jacobsthal-Lucas-Leonardo sequence is given in this paper

For more information about Jacobsthal sequence, Pell sequence, sequence, Tetranacci sequences and some generalizations of these sequences and applications of these sequences we refer to [2-5], [7-12].

The Jacobsthal-Lucas sequence $\{j_n\}$ is defined by

$$j_n = j_{n-1} + 2j_{n-2}, \quad j_0 = 2,\ j_1 = 1,\ n \geq 0. \quad (1)$$

The first values of Jacobsthal-Lucas sequence are:

2, 1, 5, 7, 17, 31, 65, 127, 257, 511, 1025, 2047, 4097, 8191.

Catarino and Borges [1] introduced Leonardo numbers. This number sequence is defined by the following recurrence relation;

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \quad (2)$$

With initial values $Le_0 = 1$ and $Le_1 = 1$. The first few values of Leonardo numbers are:

1, 1, 3, 5, 9, 15, 25, 41, 67, 109, 177, 287, 465, 753.

From [13] we have the following properties of Leonardo numbers:

a) $Le_n = 2F_{n+1} - 1$, $\quad n \geq 1$
b) $Le_{n+1} = 2Le_n - Le_{n-2}$, $\quad n \geq 2$
c) $Le_{n+1} + Le_{n-1} = 2L_{n+1} - 2$, $\quad n \geq 1$
d) $Le_n + F_n + L_n = 2Le_n + 1$, $\quad n \geq 1$
e) $Le_n + 2F_n = Le_{n+1}$. $\quad n \geq 1$

Where F_n is the nth Fibonacci number and L_n is the nth Lucas number. For more information we refer to [1] and [12].

2 JACOBSTHAL-LUCAS-LEONARDO SEQUENCE

This section is devoted to introduction of Jacobsthal-Lucas-Leonardo sequence. For convenience, we call this sequence Petroudi sequence and accept the expression (PL_n) to denote the nth term of this sequence.

Definition (2.1). The Petroudi sequence is defined by the following recurrence relation

$$PL_n = PL_{n-1} + 2PL_{n-2} + 1, \quad (2.1)$$

with initial values $PL_0 = 2$ and $PL_1 = 1$. The first few values of Petroudi sequence are;

2, 1, 6, 9, 22, 41, 86, 169, 342, 681, 1366, 2729, 5462, 10921, 21846, 43689, 87382, 174761, 349526, 699049, 1398102, 2796201, 5592406, 11184809, 22369622, 44739241, 89478486.

Remark. (2.2). From definition of Petroudi sequence we see that

$$PL_{n+1} = PL_n + 2PL_{n-1} + 1. \quad (2.2)$$

By subtraction of (2.2) and (2.1) we conclude that

$$PL_{n+1} = 2PL_n + PL_{n-1} - 2PL_{n-2} . \quad (2.3)$$

This recurrence relation has the characteristic equation $t^3 - 2t^2 - t + 2 = 0$. This equation has three distinct real roots $\alpha = 1$, $\beta = -1$ and $\lambda = 2$.

Theorem (2.3). The generating function for the Petroudi sequence(PL_n) is given as

$$\sum_{n=0}^{\infty} PL_n x^n = \frac{2 - x + 2x^2}{1 - 2x - x^2 + 2x^3}.$$

Proof. Suppose that the generating function for the Petroudi sequence (PL_n) has the formal power series

$$g(x) = \sum_{n=0}^{\infty} PL_n x^n = PL_0 + PL_1 x + PL_2 x^2 + PL_3 x^3 + \cdots + PL_n x^n + \cdots \quad .$$

Then we have

$$xg(x) = PL_0 x + PL_1 x^2 + PL_2 x^3 + PL_3 x^4 + \cdots + PL_n x^{n+1} + \cdots$$
$$x^2 g(x) = PL_0 x^2 + PL_1 x^3 + PL_2 x^4 + PL_3 x^5 + \cdots + PL_n x^{n+2} + \cdots \quad ,$$

And

$$2x^3 g(x) = 2PL_0 x^3 + 2PL_1 x^4 + 2PL_2 x^5 + 2PL_3 x^6 + \cdots + 2PL_n x^{n+3} + \cdots .$$

Thus, we obtain

$g(x) - 2xg(x) - x^2 g(x) + 2x^3 g(x) =(PL_0 + PL_1 x + PL_2 x^2 + PL_3 x^3 + \cdots + PL_n x^n + \cdots) - (2PL_0 x + 2PL_1 x^2 + 2PL_2 x^3 + 2PL_3 x^4 + \cdots + 2\, PL_n x^{n+1} + \cdots) - (PL_0 x^2 + PL_1 x^3 + PL_2 x^4 + PL_3 x^5 + \cdots + PL_n x^{n+2} + \cdots)+(2PL_0 x^3 + 2PL_1 x^4 + 2PL_2 x^5 + 2PL_3 x^6 + \cdots + 2PL_n x^{n+3} + \cdots) = (PL_0 + PL_1 x - 2PL_0 x) + (PL_2 - 2PL_1 - PL_0)x^2 + (PL_3 - 2PL_2 - PL_1 + 2PL_0)x^3 + \cdots = (2 + 1x - 2x) + (6 - 2 \times 1 - 2)x^2 + 0 = 2 - x + 2x^2.$

Therefore, by some computations we have

$$g(x) - 2xg(x) - x^2 g(x) + 2x^3 g(x) = 2 - x + 2x^2.$$

So

$$g(x)(1 - 2x - x^2 + 2x^3) = 2 - x + 2x^2.$$

Consequently

$$\sum_{n=0}^{\infty} PL_n x^n = \frac{2 - x + 2x^2}{1 - 2x - x^2 + 2x^3}.$$

Now we can describe an interesting result about the nth terms of Petroudi sequence.

Theorem (2.4). Let $n \geq 0$ be an integer. Then the Binet-like formula for the Petroudi sequence (PL_n) is

$$PL_n = \frac{1}{6}[2^{n+3} + 7(-1)^n - 3].$$

Proof. Exploiting remark (2.2), we see that the equation$f(t) = t^3 - 2t^2 - t + 2 = 0$ has three distinct roots $\alpha = 1$, $\beta = -1$ and $\gamma = 2$. Therefore the nth term of Petroudi sequence has the following form

$$PL_n = A + B(-1)^n + C\, 2^n, \quad (2.4.1)$$

Where A, B and C are constants which can be computed by initial values of Petroudi sequence$PL_0 = 2$, $PL_1 = 1$ and $PL_2 = 6$.

As $JL_0 = 2$, we obtain

$$A + B + C = 2. \quad (2.4.2)$$

As $JL_1 = 1$, we obtain

$$A - B + 2C = 1. \quad (2.4.3)$$

And as $JL_2 = 2$, we obtain

$$A + B + 4C = 6. \quad (2.4.4)$$

These linear equations form a system of linear equation. By solving this linear system, we find that $A = \frac{-1}{2}$, $B = \frac{7}{6}$ and $C = \frac{4}{3}$. By substituting the values of A, B and C in (2.4.1) we conclude that

$$PL_n = -\frac{1}{2} + \frac{7}{6}(-1)^n + \frac{4}{3}(2^n) = \frac{7(-1)^n}{6} + \frac{2^{n+2}}{3} - \frac{1}{2} = \frac{1}{6}[2^{n+3} + 7(-1)^n - 3].$$

Hence the proof is completed.

Example (2.5). For $n = 10$ we have

$$PL_{10} = \frac{1}{6}[2^{10+3} + 7(-1)^{10} - 3] = \frac{1}{6}[8192 + 7 - 3] = 1366.$$

Theorem (2.6). Let $n \geq 0$ be an integer. Then

(a) $PL_{n+1} + PL_n = 2^{n+2} - 1$.

(b) $PL_{n+1} - PL_n = \frac{1}{3}[2^{n+2} - 7(-1)^n]$.

(c) $PL_{2n} = \frac{2}{3}(2^{2n+1} + 1)$.

(d) $PL_{2n+1} = \frac{1}{3}(2^{2n+3} - 5)$.

Proof. We prove part(a). Other identities, similarly can be proved. Using theorem (2.4) we have

$$\begin{aligned} PL_{n+1} + PL_n &= \frac{1}{6}[2^{n+4} + 7(-1)^{n+1} - 3] + \frac{1}{6}[2^{n+3} + 7(-1)^n - 3] \\ &= \frac{1}{6}[4(2^{n+2}) - 7(-1)^n - 3 + 2(2^{n+2}) + 7(-1)^n - 3] = \frac{6(2^{n+2}) - 6}{6} = 2^{n+2} - 1. \end{aligned}$$

Example (2.7). Correspond to identities of the last theorem, for $n = 12$ we have

$PL_{13} + PL_{12} = 10921 + 5462 = 16383 = 16384 - 1 = 2^{14} - 1$,

$PL_{13} - PL_{12} = 10921 - 5462 = 5459 = \frac{1}{3}(16384 - 7) = \frac{1}{3}[2^{14} - 7(-1)^{12}]$,

$PL_{24} = \frac{2}{3}(2^{25} + 1) = 22369622$,

$PL_{25} = PL_{24+1} = \frac{1}{3}(2^{24+3} - 5) = 44739241$.

Theorem (2.8). Let $n \geq 0$ be an integer and k be an arbitrary integer. Then

(a) $PL_{n+k} + PL_{n-k} = \frac{4}{3}[2^{n+k} + 2^{n-k}] + \frac{7}{6}[(-1)^{n+k} + (-1)^{n-k}] - 1.$

(b) $PL_{n+k} - PL_{n-k} = \frac{4}{3}[2^{n+k} - 2^{n-k}] + \frac{7}{6}[(-1)^{n+k} - (-1)^{n-k}].$

Proof. They can be proved directly, by using the definition of Petroudi sequence.

Corollary (2.9). Let $n \geq 0$ be an integer. Then we have

(a) $PL_{n+1} + PL_{n-1} = \frac{10}{3}(2^n) - \frac{7}{3}(-1)^n - 1,$

(b) $PL_{n+1} - PL_{n-1} = 2^{n+1}.$

Proof. They can be proved by substituting $k = 1$ in theorem (2.8).

Example (2.10). For n=11 we have

$6828 = 1366 + 5462 = PL_{12} + PL_{10} = \frac{10}{3}(2^{11}) - \frac{7}{3}(-1)^{11} - 1,$

$4096 = 5462 - 1366 = PL_{12} - PL_{10} = 2^{11+1}.$

Theorem (Cassini's identity) (2.11). Let $n \geq 0$ be an integer. Then

$$PL_{n+1} \times PL_{n-1} - (PL_n)^2 = \frac{7(-1)^n - 21(-2)^n - 2^n}{3}.$$

Proof. Using theorem (2.4), one can prove it, by direct calculations.

Example (2.12). For $n = 9$ we have

$$PL_{10} \times PL_8 - (PL_9)^2 = 1366 \times 342 - (681)^2 = 3411 = \frac{7(-1)^9 - 21(-2)^9 - 2^9}{3}.$$

3 SUMMATION FORMULA S

In this section we establish some summation formulas for the Petroudi sequence.

Lemma (3.1). [7] Let $n \geq 0$ be an integer. Then we have

$$\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x-1}, \tag{10}$$

$$\sum_{k=0}^{n-1} k x^k = \frac{(n-1)x^n - n x^{n-1} + 1}{(x-1)^2}. \tag{11}$$

Theorem (3.2). Let $n \geq 0$ be an integer. Then

$$(a) \sum_{k=0}^{n} PL_k = \frac{1}{12}[2^{n+5} + 7(-1)^n - 6n - 15].$$

$$(b) \sum_{k=0}^{2n} PL_k = \frac{1}{3}(2^{2n+3} - 3n - 2).$$

$$(c) \sum_{k=0}^{2n+1} PL_k = \frac{1}{3}(2^{2n+4} - n - 7)$$

Proof. Exploiting theorem (2.4) and summation formulas of (10), we obtain

$$\sum_{k=0}^{n} PL_k = \sum_{k=0}^{n} \frac{1}{6}[2^{k+3} + 7(-1)^k - 3] = \frac{1}{6}\left(\sum_{k=0}^{n}(2^{k+3}) + \sum_{k=0}^{n} 7(-1)^k - \sum_{k=0}^{n} 3\right)$$
$$= \frac{1}{6}\left(8\sum_{k=0}^{n} 2^k + 7\sum_{k=0}^{n}(-1)^k - 3\sum_{k=0}^{n} 1\right)$$
$$= \frac{1}{6}\left(\left[\frac{8(2^{n+1}-1)}{2-1}\right] + 7\left[\frac{(-1)^{n+1}-1}{-1-1}\right] - 3(n+1)\right)$$
$$= \frac{1}{12}[2^{n+5} + 7(-1)^n - 6n - 15].$$

This equality completes the proof of part (a). Other summation identities can be proved by similar manner.

Example (3.3). For n=23 we find that

$$\sum_{k=0}^{23} PL_k = 22369608, \quad \sum_{k=0}^{46} PL_k = 187649984473747, \quad \sum_{k=0}^{47} PL_k = 375299968947516.$$

Theorem (3.4). Let $k \geq 0$ be an integer. Then

$$\sum_{k=0}^{n} 2^{k+1} \times PL_k = \frac{1}{9}[2^{2n+4} + 7(-2)^n + 4] - 2^n.$$

Proof. It can be proved by similar method which we used in theorem (3.2).

The next theorem shows an interesting relation between theorem (3.4) and product of two consecutive terms of Petroudi sequence PL_n.

Theorem (3.5). Let $k > 0$ be an integer. Then

$$\sum_{k=1}^{n} 2^{k+1} \times PL_k = PL_n \times PL_{n+1} - 2. \qquad \text{(I)}$$

Proof. We prove this theorem by mathematical induction on n.

For $n = 1$ we have $\sum_{k=1}^{1} 2^{k+1} PL_k = 2^2 PL_1 = 4 \times 1 = 6 - 2 = PL_1 \times PL_2 - 2$. Hence both sides are equal and (I) is true for $n = 1$.

Now suppose that (I) is true for $n = k$. In exact suppose $\sum_{m=1}^{k} 2^{m+1} PL_m = PL_k \times PL_{k+1} - 2$ is true. Then

$$\sum_{m=1}^{k+1} 2^{m+1} \times PL_m = \sum_{m=1}^{k} 2^{m+1} PL_m + 2^{k+2} PL_{k+1} = PL_k \times PL_{k+1} - 2 + 2^{k+2} PL_{k+1}$$
$$= PL_{k+1}(PL_k + 2^{k+2}) - 2.$$

Using theorem (2.6) (a) we know that $PL_{k+1} + PL_k = 2^{k+2} - 1$. Thus, we find that

$$\sum_{m=1}^{k+1} 2^m \times PL_m = PL_{k+1}(PL_k + PL_{k+1} + PL_k + 1) - 2 = PL_{k+1}(PL_{k+1} + 2PL_k + 1) - 2$$
$$= PL_{k+1} \times PL_{k+2} - 2.$$

Thus, (I) is true for $n = k + 1$. Consequently, by the principle of Mathematical induction, (I) is true for all positive integers n.

Theorem (3.6). Let $k \geq 0$ be an integer. Then

$$(a)\sum_{k=0}^{n} PL_{2k} = \frac{2}{9}(2^{2n+3} + 3n + 1).$$

$$(b)\sum_{k=0}^{2n} PL_{2k} = \frac{2}{9}(2^{4n+3} + 6n + 1).$$

$$(c)\sum_{k=0}^{2n+1} PL_{2k} = \frac{2}{9}(2^{4n+5} + 6n + 4).$$

$$(d)\sum_{k=0}^{n} PL_{2k+1} = \frac{1}{9}(2^{2n+5} - 15n - 23).$$

$$(e)\sum_{k=0}^{2n} PL_{2k+1} = \frac{1}{9}(2^{4n+5} - 30n - 23).$$

$$(f)\sum_{k=0}^{2n+1} PL_{2k+1} = \frac{2}{9}(2^{4n+6} - 15n - 9).$$

Proof. Exploiting theorem (2.4), and using summation formula of (10) one can prove these summation formulas by similar method, which we used to prove theorem (3.2).

Theorem (3.7). Let $k \geq 0$ be an integer. Then

$$(a)\sum_{k=0}^{n-1} k \times PL_k = -\frac{1}{24}[(14n-7)(-1)^n - 2^{n+5}(n-2) + 6n(6n-1) - 57].$$

$$(b)\sum_{k=0}^{n-1} k \times PL_{2k} = \frac{1}{27}[2^{2n}(12n-16) + 9n(9n-1) + 16].$$

$$(c)\sum_{k=0}^{n-1} k \times PL_{2k+1} = \frac{1}{54}[2^{2n}(48-64) - 45n(n-1) + 64].$$

Proof. They can be proved by direct calculation using summation formula of (11) and theorem (2.4).

Theorem (3.8). Let $n \geq 0$ be an integer. Then

$$(a)\sum_{k=0}^{n-1} PL_k \times PL_{k+1} = \frac{2}{27}[2^{2n+4} - 27(2^n) - 7(-2)^n - 15n + 18].$$

$$(b)\sum_{k=0}^{n-1} PL_{k-1} \times PL_{k+1} = \frac{1}{108}[2^{2n+6} - 180(2^n) + 140(-2)^n - 63(-1)^n + 174n + 39].$$

Proof. It can be proved by direct calculation using summation formula of (10), (11) and theorem (2.4).

4 PYHTON CODE

In this section, we give PYTHON code to generate the first n terms of Petroudi sequence. In order to performing this code, we used online PYTHON compiler.

```
# Program to display the Petroudi sequence up to n-th term
nterms = int(input("How many terms would you like to display? "))# first two terms
n1, n2 = 2, 1
count = 0
# the number of terms must be valid. thus check if the number of terms is valid.
if nterms <= 0:
   print("It is note correct. Please enter a positive integer")
# if there is only one term, return n1
elif nterms == 1:
   print("Petroudi sequence upto",nterms,":")
   print(n1)
# generate Petroudi sequence
else:
   print("Petroudi sequence:")
   while count < nterms:
       print(n1)
       nth =n2+2*n1+1
       # this section update values
       n1 = n2
       n2 = nth
       count += 1
```

5 CONCLUSION

In this paper we introduced, Jacobsthal-Lucas-Leonardo sequence (Petroudi sequence). We represented the Binet-like formulas and generating function of this sequence. We obtained some interesting identities and summation formulas about this sequence. Moreover, we gave some examples about these identities and summation formulas. Also, we proposed PYTHON code to generate the first n terms of this, using online PYTHON compiler. For the future work, one can prove summation formulas about the combination of Padovan sequence, Perrin sequence and Narayana sequence with Leonardo sequence. Also, we can consider some matrices involving Petroudi sequence and investigate their spectral norm properties.

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