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Vertex decomposability path complexes of trees

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Abstract

A tree is called double starlike if it has exactly two vertices of degree greater than two. Let $H(p,n,q)$ denote the double starlike tree obtained by attaching p pendant vertices to one pendant vertex of the path Pn and q pendant vertices to the other pendant vertex of Pn. Also let $H(p,n)$ be graph obtained by attaching p pendant vertices to one pendant vertex of the path Pn. Let G be an undirected tree. We prove that $\Delta_t\,(G)$ is vertex decomposable for all $t \geq 2$ if and only if $G = H(p,n,q)$ or $G = H(p,n)$.

Keywords: Vertex decomposable, simplicial complex, Shellable

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1 Introduction

Let $R = K[x_1, \ldots, x_n]$, where K is a field. Fix an integer $n \geq t \geq 2$ and let G be an undirected graph. A sequence $x_{i_1}, \ldots, x_{i_t}$ of distinct vertices is called a **path** of length t if there are $t-1$ distinct edges $e_1, \ldots, e_{t-1}$ where e_j is a edge from x_{i_j} to $x_{i_{j+1}}$ or from $x_{i_{j+1}}$ to x_{i_j} . Then the path ideal of G of length t is the monomial ideal $I_t(G) = (x_{i_1} \ldots x_{i_t} : x_{i_1}, \ldots, x_{i_t}$ is a path of length t in G)
in the polynomial ring $R = K[x_1, \ldots, x_n]$. The distance $d(x,y)$ of two vertices x and y of a graph G is the length of the shortest path from x to y. Also we define the simplicial complex $\Delta_t(G)$ to be

$$\Delta_t(G) = \langle \{x_{i_1}, \ldots, x_{i_t}\} : x_{i_1}, \ldots, x_{i_t} \text{ is a path of length t in G } \rangle.$$

Path ideals of graphs were first introduced by Conca and De Negri [5] in the context of monomial ideals of linear type. In [6] it has been shown that, $\Delta_t(G)$ is a simplicial tree if G is a rooted tree and $t \geq 2$. Ajdani and Bulnes in [1] proved vertex decomposability path complexes of cycles. In this paper, we focus on the path complexes of trees. Throughout the paper, we mean by tree, an undirected tree and by a path, an undirected path. This paper is organized as follows. In next Section we recall several definitions and terminology which we need later. In Section 3, for all $t \geq 2$ we show that $\Delta_t(G)$ is vertex decomposable if and only if $G = H(p,n,q)$ or $G = H(p,n)$.

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2 Preliminaries

In this section we recall some definitions and results which will be needed later.

Definition 2.1. A *simplicial complex* Δ over a set of vertices $V = \{x_1, \ldots, x_n\}$, is a collection of subsets of V, with the property that:

(a) $\{x_i\} \in \Delta$, for all i;

(b) if $F \in \Delta$, then all subsets of F are also in Δ (including the empty set).

An element of Δ is called a *face* of Δ and complement of a face F is $V \setminus F$ and it is denoted by F^c. Also, the complement of the simplicial complex $\Delta = \langle F_1, \ldots, F_r \rangle$ is $\Delta^c = \langle F_1^c, \ldots, F_r^c \rangle$. The *dimension* of a face F of Δ, $\dim F$, is $|F| - 1$ where, $|F|$ is the number of elements of F. A *non-face* of Δ is a subset F of V with $F \notin \Delta$. we denote by $\mathcal{N}(\Delta)$, the set of all minimal non-faces of Δ. The maximal faces of Δ under inclusion are called *facets* of Δ. The *dimension* of the simplicial complex Δ, $\dim \Delta$, is the maximum of dimensions of its facets. If all facets of Δ have the same dimension, then Δ is called *pure*. Let $\mathcal{F}(\Delta) = \{F_1, \ldots, F_q\}$ be the facet set of Δ. It is clear that $\mathcal{F}(\Delta)$ determines Δ completely and we write $\Delta = \langle F_1, \ldots, F_q \rangle$. A simplicial complex with only one facet is called a *simplex*. A simplicial complex Γ is called a *subcomplex* of Δ, if $\mathcal{F}(\Gamma) \subset \mathcal{F}(\Delta)$.

For $v \in V$, the subcomplex of Δ obtained by removing all faces $F \in \Delta$ with $v \in F$ is denoted by $\Delta \setminus v$. That is,

$$\Delta \setminus v = \langle F \in \Delta : \quad v \notin F \rangle.$$

The *link* of a face $F \in \Delta$, denoted by $\operatorname{link}_\Delta(F)$, is a simplicial complex on V with the faces, $G \in \Delta$ such that, $G \cap F = \varnothing$ and $G \cup F \in \Delta$. The link of a vertex $v \in V$ is simply denoted by $\operatorname{link}_\Delta(v)$.

$$\operatorname{link}_\Delta(v) = \big\{F \in \Delta : \qquad v \notin F, \quad F \cup \{v\} \in \Delta\big\}.$$

Definition 2.2. Let Δ be a simplicial complex over n vertices $\{x_1, \ldots, x_n\}$. For $F \subset \{x_1, \ldots, x_n\}$, we set:

$$\mathbf{x}_F = \prod_{x_i \in F} x_i.$$

We define the *facet ideal* of Δ, denoted by $I(\Delta)$, to be the ideal of S generated by $\{\mathbf{x}_F : \quad F \in \mathcal{F}(\Delta)\}$. The *non-face ideal* or the *Stanley-Reisner ideal* of Δ, denoted by I_Δ, is the ideal of S generated by square-free monomials $\{\mathbf{x}_F : \quad F \in \mathcal{N}(\Delta)\}$. Also we call $K[\Delta] := S/I_\Delta$ the *Stanley-Reisner ring* of Δ.

Definition 2.3. A simplicial complex Δ is recursively defined to be *vertex decomposable*, if it is either a simplex, or else has some vertex v so that,

(a) Both $\Delta \setminus v$ and $\operatorname{link}_\Delta(v)$ are vertex decomposable, and

(b) No face of $\operatorname{link}_\Delta(v)$ is a facet of $\Delta \setminus v$.

A vertex v which satisfies in condition (b) is called a *shedding vertex*.

Definition 2.4. A simplicial complex Δ is *shellable*, if the facets of Δ can be ordered $F_1, \ldots, F_s$ such that, for all $1 \leq i < j \leq s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{v\}$.

Definition 2.5. A graded S-module M is called *sequentially Cohen-Macaulay* (over K), if there exists a finite filtration of graded S-modules,

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each M_i/M_{i-1} is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

A simplicial complex Δ is called (sequentially) Cohen-Macaulay over K, if the ring $K[\Delta] = S/I_\Delta$ is (sequentially) Cohen-Macaulay. A simplicial complex Δ is called disconnected, if the vertex set V of Δ is a disjoint union $V = V_1 \cup V_2$ such that no face of Δ has vertices in both V_1 and V_2. Otherwise Δ is connected.

Remark 2.6. All Cohen-Macaulay simplicial complexes of positive dimension are connected.

Lemma 2.7. *Let $\Delta_t(Pn)$ be a simplicial complex on the vertices $\{x_1, \dots, x_n\}$ and $Pn = x_1, \dots, x_n$. Then $\Delta_t(Pn)$ is vertex decomposable for all $t \geq 2$.*

Proof. If $t = n$, then $\Delta_n(Pn)$ is a simplex which is vertex decomposable. Let $2 \leq t < n$ then one has

$$\Delta_t(Pn) = \langle \{x_1, \dots, x_t\}, \{x_2, \dots, x_{t+1}\}, \dots, \{x_{n-t+1}, \dots, x_n\} \rangle.$$

So $\Delta_t(Pn) \setminus x_n = \langle \{x_1, \dots, x_t\}, \{x_2, \dots, x_{t+1}\}, \dots, \{x_{n-t}, \dots, x_{n-1}\} \rangle$. Now we use induction on the number of vertices of Pn and by induction hypothesis $\Delta_t(Pn) \setminus x_n$ is vertex decomposable. On the other hand, it is clear that $\mathrm{link}_{\Delta_t(Pn)}\{x_n\} = \langle \{x_{n-t+1}, \dots, x_{n-1}\} \rangle$. Thus $\mathrm{link}_{\Delta_t(Pn)}\{x_n\}$ is a simplex which is not a facet of $\Delta_t(Pn) \setminus x_n$. Therefore $\Delta_t(Pn)$ is vertex decomposable. □

3 Characterization of path complexes of trees

As the main result of this section, for all $t \geq 2$, we characterize all such trees whose $\Delta_t(G)$ is vertex decomposable. Let $H(p, n, q)$ denote the double starlike tree obtained by attaching p pendant vertices to one pendant vertex of the path Pn and q pendant vertices to the other pendant vertex of Pn. Also let $H(p, n)$ be graph obtained by attaching p pendant vertices to one pendant vertex of the path Pn.

Remark 3.1. Let $Pn = x_1, \dots, x_n$ be a path on vertices $\{x_1, \dots, x_n\}$ and $H(2, n)$ be a graph obtained by attaching two pendant vertices to pendant vertex x_n. Then $\Delta_t(H(2, n))$ is vertex decomposable for all $t \geq 2$.

Proof. By lemma 2.7 proof is trivial. □

Proposition 3.2. *Let $Pn = x_1, \dots, x_n$ be a path on vertices $\{x_1, \dots, x_n\}$ and $H(p, n)$ be a graph obtained by attaching p pendant vertices to pendant vertex x_n. Then $\Delta_t(H(p, n))$ is vertex decomposable for all $t \geq 2$.*

Proof. We prove the proposition by induction on p the number of pendant vertices to pendant vertex x_n of Pn. If $p = 0$ or 1 then $H(p, n)$ is a path and by lemma 2.7 $\Delta_t(H(p, n))$ is vertex decomposable. If $p = 2$ then by remark 3.1 $\Delta_t(H(p, n))$ is vertex decomposable. Now let $p > 2$ and $\{y_1, \dots, y_p\}$ be p pendant vertices to pendant vertex x_n of Pn then one has

$$H(p, n) \setminus \{y_1\} = H(p - 1, n)$$

and

$$\Delta_t(H(p, n)) \setminus \{y_1\} = \Delta_t(H(p - 1, n)).$$

Therefore by induction hypothesis $\Delta_t(H(p - 1, n))$ is vertex decomposable. So $\Delta_t(H(p, n)) \setminus \{y_1\}$ is vertex decomposable. If $t = 3$ then we have

$$\mathrm{link}_{\Delta_3(H(p,n))}\{y_1\} = \langle \{x_{n-1}, x_n\}, \{y_2, x_n\}, \dots, \{y_p, x_n\} \rangle.$$

It is easy to see that $\mathrm{link}_{\Delta_3(H(p,n))}\{y_1\}$ is vertex decomposable and y_1 is a shedding vertex. If $t = 2$ or $t > 3$, one has

$$\mathrm{link}_{\Delta_t(H(p,n))}\{y_1\} = \langle \{x_{n-t+2}, \dots, x_n\} \rangle.$$

Thus $\mathrm{link}_{\Delta_t(H(p,n))}\{y_1\}$ is a simplex, which is not a facet of $\Delta_t(H(p, n)) \setminus \{y_1\}$, therefore $\Delta_t(H(p, n))$ is vertex decomposable. □

Lemma 3.3. *Let $p = 2$ and $q \geq 2$, Then $\Delta_t(H(2, n, q))$ is vertex decomposable for all $2 \leq t \leq n + 2$*

Proof. Let $H(2,n,q)$ denote the double starlike tree obtained by attaching two pendant vertices $\{y_1,y_2\}$ to pendant vertex x_1 of path Pn and $\{y'_1,\ldots,y'_q\}$ be pendant vertices to pendant vertex x_n of Pn. So by proposition 3.2 $\Delta_t(H(2,n,q))\setminus\{y_1\}$ is vertex decomposable. Now we prove that $\mathrm{link}_{\Delta_t(H(2,n,q))}\{y_1\}$ is vertex decomposable. If $t=3$ then $\mathrm{link}_{\Delta_3(H(2,n,q))}\{y_1\}=\langle\{x_1,x_2\},\{x_1,y_2\}\rangle$ which is vertex decomposable. If $t=n+2$ then

$$\mathrm{link}_{\Delta_{n+2}(H(2,n,q))}\{y_1\}=\langle\{x_1,\ldots,x_n,y'_1\},\{x_1,\ldots,x_n,y'_2\},\ldots,\{x_1,\ldots,x_n,y'_q\}\rangle.$$

It is easy to see that $\mathrm{link}_{\Delta_{n+2}(H(2,n,q))}\{y_1\}$ is vertex decomposable. If $t=2$ or $4\le t\le n+1$ then we have $\mathrm{link}_{\Delta_t(H(2,n,q))}\{y_1\}=\langle\{x_1,\ldots,x_{t-1}\}\rangle$. Thus $\mathrm{link}_{\Delta_t(H(2,n,q))}\{y_1\}$ is a simplex which is vertex decomposable. It is clear that y_1 is a shedding vertex. □

Proposition 3.4. *Let Q_1,Q_2 be two paths of maximum length k in tree G and y be a leaf of G such that $y\in Q_1\cap Q_2$, $|Q_1\cap Q_2|=L$. Then $\Delta_k(G)$ is not vertex decomposable.*

Proof. Suppose $Q_1=y_1,y_2,\ldots,y_{k-L},x_1,x_2,\ldots,x_{L-1},y$ and
$Q_2=y'_1,y'_2,\ldots,y'_{k-L},x_1,x_2,\ldots,x_{L-1},y$ be two paths of length k in G such that $Q_1\cap Q_2=\{x_1,x_2,\ldots,x_{L-1},y\}$ and $deg(y)=1$. Since $\mathrm{link}_{\Delta_k(G)}\{x_1,\ldots,x_{L-1},y\}$ is disconnected and pure of positive dimension. By remark 2.6 $\Delta_k(G)$ is not Cohen-Macaulay and hence $\Delta_k(G)$ is not vertex decomposable. □

Proposition 3.5. *Let G be a double starlike tree such that $G=H(p,n,q)$. Then $\Delta_t(G)$ is vertex decomposable for all $2\le t\le n+2$.*

Proof. Let $G=H(p,n,q)$ denote the double starlike tree obtained by attaching p pendant vertices to one pendant vertex of the path Pn and q pendant vertices to the other pendant vertex of Pn. We prove the theorem by induction on p the number of pendant vertices to pendant vertex x_1 of Pn. If $p=0$ or $p=1$ then by proposition 3.2 $\Delta_t(G)$ is vertex decomposable. If $p=2$ then by lemma 3.3 $\Delta_t(G)$ is vertex decomposable. Now let $p>2$ and $\{y_1,\ldots,y_p\}$ be p pendant vertices to pendant vertex x_1 of Pn. Since $G\setminus\{y_1\}$ is again double starlike tree on $p-1$ pendant vertices. Therefore by induction hypothesis $\Delta_t(G\setminus\{y_1\})$ is vertex decomposable. So $\Delta_t(G\setminus\{y_1\})=\Delta_t(G)\setminus\{y_1\}$ is vertex decomposable. Let $t=2$ then $\mathrm{link}_{\Delta_2(G)}\{y_1\}=\langle\{x_1\}\rangle$ is simplex and vertex decomposable. Let $t=3$ then $\mathrm{link}_{\Delta_3(G)}\{y_1\}=\langle\{x_2,x_1\},\{y_2,x_1\},\ldots,\{y_p,x_1\}\rangle$ is vertex decomposable. Let $3<t\le n+1$ then $\mathrm{link}_{\Delta_t(G)}\{y_1\}=\langle\{x_1,x_2,\ldots,x_{t-1}\}\rangle$ is simplex and vertex decomposable. Let $t=n+2$ then $\mathrm{link}_{\Delta_t(G)}\{y_1\}=\langle\{x_1,\ldots,x_n,y_1\},\{x_1,\ldots,x_n,y_2\},\ldots,\{x_1,\ldots,x_n,y_p\}\rangle$ is a path complex of a starlike tree which is vertex decomposable. It is easy to see that no face of $\mathrm{link}_{\Delta_t(G)}\{y_1\}$ is a facet of $\Delta_t(G)\setminus\{y_1\}$. So $\Delta_t(G)$ is vertex decomposable. □

Now, we are ready that prove the main result of this paper.

Theorem 3.6. *Let G be a tree such that is not a path. Then $\Delta_t(G)$ is vertex decomposable for all $t\ge 2$ if and only if $G=H(p,n,q)$ or $G=H(p,n)$.*

Proof. ($\Longrightarrow$)We prove by contradiction. Suppose $G\neq H(p,n,q)$ and $G\neq H(p,n)$. So there exists two paths of maximum length k in G which contain L common vertices such that one of these vertices is a leaf. Therefore by proposition 3.4 $\Delta_k(G)$ is not vertex decomposable which is a contradiction. ($\Longleftarrow$) By proposition 3.2 and proposition 3.5 the proof is completed. □

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