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On normal Cayley hypergraphs of finite groups

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Abstract

We call a Cayley hypergraph $X = CH(G, S)$ normal for G if G_R, the right regular representation of G, is a normal subgroup of the full automorphism group $Aut(X)$ of X. In this paper, we introduce the concept of normal t-Cayley hypergraphs and some related problems.

Keywords: Hypergraph, Cayley hypergraph, Normal t-Cayley hypergraph

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1 Introduction

A *hypergraph* X on a finite set V is a family $E = (E_i)_{i\in I}$, $I = \{1, 2, ..., n\}$, $n \in N^*$ of nonempty subsets of V called *hyperedges* with $\bigcup_{i\in I} E_i = V$. Let us denote $X = (V, (E_i)_{i\in I}) = (E_1, E_2, ..., E_m)$. The elements of V are called *vertices* and the ones of the family $(E_i)_{i\in I}$ are called *hyperedges*. The cardinality of the set of vertices is called the *order* of the hypergraph. The set of vertices, the family of edges and the set of indices of a hypergraph X are denoted by $V(X)$, $E(X)$ and $I(X)$, respectively. Let $u, v \in V(X)$, and $E_i \in E(X)$ where $i \in I$. Then we say that E_i is *incident* with u when $u \in E_i$ and that two distinct u, v are *adjacent* when there is an edge incident with both u and v. The family of edges $E(v) = E_X(v) \subseteq E(X)$ incident to a vertex v is called the *star* of v. The *degree* $d(v) = d_X(v)$ of a vertex v is the number of edges of its star $E(v)$. A hypergraph in which all vertices have the same degree d is said to be *regular* of degree d or *d-regular*. A hypergraph X is called *k-uniform* or a *k-hypergraph* if every edge has cardinality k. We note that a graph is just a 2-uniform hypergraph. For two given hypergraphs X_1=(V_1, E_1) and X_2=(V_2, E_2), a *hypergraph isomorphism* $\varphi : X_1 \to X_2$ is a bijection $\varphi : V_1 \to V_2$ such that e is an edge in X_1 if and only if $\varphi(e)$ is an edge in X_2. An isomorphism from a hypergraph X onto itself is an *automorphism*. The *automorphism group* of X is denoted by $Aut(X)$. For more information about hypergraphs, we refer the reader to [3].

Let G be a finite group and let S be a subset of $G - \{1_G\}$, where 1_G is the identity element in G and $S = S^{-1}$ is inverse closed. The *Cayley graph* $X = Cay(G, S)$ of G with respect to S is defined as the graph with vertex set $V(X) = G$, and edges the pairs $\{g, h\}$ for which $hg^{-1} \in S$. A Cayley graph $Cay(G, S)$ is connected if and only if $G = \langle S\rangle$, that is, S generates G. It is well known that $Aut(X)$ contains the right regular representation G_R of G, the acting group of G by right multiplication, which is regular on vertex set $V(X)$. Thus Cayley graphs are vertex-transitive. Also, we note that the group $Aut(G, S) := \{\alpha \in Aut(G)|S^\alpha = S\}$ is a subgroup of $Aut(X)_1$, the stabilizer of the vertex 1_G in $Aut(X)$. The Cayley graph $X = Cay(G, S)$ is said to be *normal* if $G_R \lhd Aut(X)$. Let $N_{Aut(X)}(G_R)$ be the normalizer of G_R in $Aut(X)$. By Godsil [5], $N_{Aut(X)}(G_R) = G_R : Aut(G, S)$, the semidirect product of G_R and $Aut(G, S)$. Thus, a Cayley graph X is normal if and only if its automorphism group is the semidirect product of G_R and $Aut(G, S)$.

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Let G be a group and let S be a set of subsets $s_1, s_2, ..., s_n$ of $G - \{1_G\}$ such that $G = \langle \bigcup_{i=1}^{n} s_i \rangle$, that is, $\bigcup_{i=1}^{n} s_i$ generate G. A *Cayley hypergraph* $CH(G, S)$ has vertex set G and edge set $\{\{g, gs\} | g \in G, s \in S\}$, where an edge $\{g, gs\}$ is the set $\{g\} \cup \{gx | x \in s\}$. For all $s \in S$, if $|s| = 1$, then the Cayley hypergraph is a Cayley graph. For example, we consider the hypergraph X, with $V(X) = \{0, 1, 2, 3, 4, 5, 6\}$,

$$E(X) = \{\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 0\}, \{5, 6, 1\}, \{6, 0, 2\}\}.$$

This hypergraph which is called the Fano plane, is the Cayley hypergraph $X = CH(\mathbb{Z}_7, \{1, 3\})$. The hypergraph of the Fano plane is the 3-uniform hypergraph, which is the unique triple system with 7 hyperedges on 7 vertices where every pair of vertices is contained in precisely one hyperedge (Figure 1).

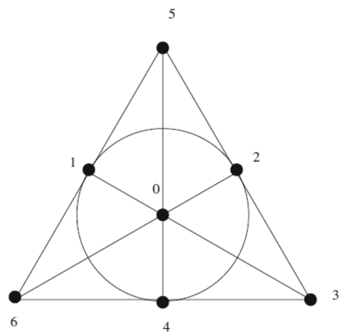

Figure 1: The hypergraph of Fano plane

The concept of normality of the Cayley graph is known to be of fundamental importance for the study of arc transitive graphs. So, for a given finite group G, a natural problem is to determine all the normal or non-normal Cayley graph of G. Some meaningful results in this direction, especially for the undirected Cayley graphs, have been obtained. Baik et al. [1] determined all non-normal Cayley graphs of abelian groups with valency at most 4 and later [2] dealt with valency 5. For directed Cayley graphs, Xu et al. [7] determined all non-normal Cayley graphs of abelian groups with valency at most 3.

In section 2, we introduce the concept of t-Cayley hypergraphs and we give some related concepts and results.

2 Main results

In this section we introduce the concept of t-Cayley hypergraphs and and some related problems.

Definition 2.1. Let G be a finite group, S a subset of $G - \{1_G\}$ and t an integer satisfying $2 \leq t \leq \max\{o(s) | s \in S\}$. The *$t$-Cayley hypergraph* $X = t\text{-}Cay(G, S)$ of G with respect to S is defined by: $V(X) := G$, and for $E \subseteq G$, $E \in E(X)$ if and only if $\exists g \in G$, $\exists s \in S$: $E = \{gs^i | 0 \leq i \leq t-1\}$.

Note that any 2-Cayley hypergraph (of G with respect to S) is a Cayley graph(of G with respect to S) and vice versa. For any $s_i \in S$, if $s_i = \{s, ..., s^{t-1}\}$ for some $s \in G - \{1_G\}$, then the Cayley hypergraph $CH(G, S)$ is a t-Cayley hypergraph $t - Cay(G, S)$. Hence a Cayley hypergraph is a generalization of a t-Cayley hypergraph.

Lemma 2.2. *Let $X = t\text{-}Cay(G, S)$ be a t-Cayley hypergraph where S is a subset of $G - \{1_G\}$. Then $Aut(G) \cap Aut(X) = Aut(G, S)$.*

Proof. By definition we have $Aut(G, S) = \{\alpha \in Aut(G), S^\alpha = S\}$. Let now $\alpha \in Aut(X) \cap Aut(G)$. We claim that $S^\alpha = S$. Now $s \in S$ if and only if $\{1, s, s^2, s^3, ..., s^{t-1}\} \in E(X)$ if and only if $\{1, s, s^2, s^3, ..., s^{t-1}\}^\alpha \in E(X)$ if and only if $\{1 = 1^\alpha, s^\alpha, (s^2)^\alpha, ..., (s^{t-1})^\alpha\} \in E(X)$ if and only if $s^\alpha \in S$, therefore $S^\alpha = S$ and hence $\alpha \in Aut(G, S)$. So $Aut(G) \cap Aut(X) \leq Aut(G, S)$. Now assume $\alpha \in Aut(G, S)$ which by definition means that $\alpha \in Aut(G)$. We will have $e \in E(X)$ if and only if $\exists s \in$ S such that $e = \{x, xs, xs^2, ..., xs^{t-1}\} \in E(X)$

if and only if $\{x^\alpha, x^\alpha s^\alpha, x^\alpha (s^2)^\alpha, ..., x^\alpha (s^{t-1})^\alpha\} \in E(X)$ if and only if $\{x^\alpha, x^\alpha s', x^\alpha (s')^2, ..., x^\alpha (s')^{t-1}\} \in E(X)$, where $s^\alpha = s'$. Thus $\alpha \in Aut(X)$ and so $\alpha \in Aut(X) \cap Aut(G)$, which implies $Aut(G,S) \leq Aut(X) \cap Aut(G)$. □

Now we have the following results from Lemma 2.2. Write $A := Aut(X)$.

Lemma 2.3. *Let $X = t\text{-}Cay(G,S)$ be a t-Cayley hypergraph of G with respect to S. Then $N_A(G_R) = G_R : Aut(G,S)$. Furthermore, the stabilizer of 1_G in $N_A(G_R)$ is $Aut(G,S)$.*

Definition 2.4. Let $X = t\text{-}Cay(G,S)$ be a t-Cayley hypergraph of G with respect to S. Then X is called *normal* if $G_R \trianglelefteq A$.

The following obvious result is a direct consequence of the above definition and Lemma 2.2.

Lemma 2.5. *Let $X = t\text{-}Cay(G,S)$. Then X is normal if and only if $A_1 = Aut(G,S)$, where A_1 is the stabilizer of 1_G in A.*

Proposition 2.6. *Let G be a finite group. S a generating set of G not containing the identity 1, and α an automorphism of G. Then t-Cayley hypergraph $X = t\text{-}Cay(G,S)$ is normal if and only if $X' = t\text{-}Cay(G,S^\alpha)$ is normal.*

Proof. Let $A' = Aut(X')$. We prove that (1) $\alpha^{-1}A\alpha = A'$, and (2) $\alpha^{-1}G_R\alpha = G_R$. For the first equation, we suppose that $\alpha^{-1}\rho\alpha \in \alpha^{-1}A\alpha$, where $\rho \in A$. Now if $E' \in E(X')$, then $E' = \{xs^i | 0 \leq i \leq t-1\}$ for some $x \in G$ and $s \in S$. Therefore $(E')^{\alpha^{-1}\rho\alpha} = \{(xs^i)^{\alpha^{-1}\rho\alpha} | 0 \leq i \leq t-1\} = \{x^{\alpha^{-1}}, x^{\alpha^{-1}}(s)^{\alpha^{-1}}, ..., x^{\alpha^{-1}}(s^{t-1})^{\alpha^{-1}}\}^{\rho\alpha}$. It follows that,
$(E')^{\alpha^{-1}\rho\alpha} = \{y, ys', y(s')^2, ..., y(s')^{t-1}\}^{\rho\alpha}$, where $s' = s^{\alpha^{-1}}$ and $x^{\alpha^{-1}} = y$. Since $\rho \in A$, $(E')^{\alpha^{-1}\rho\alpha} = \{z, zs'', ..., z(s'')^{t-1}\}^{\rho} \in E(X')$, where $s'' = (s')^\alpha$ and $y^\alpha = z$. With the similar argument $A' \subseteq \alpha^{-1}A\alpha$ and so $\alpha A \alpha^{-1} = A'$. Also it is easy to see that $\alpha^{-1}G_R\alpha = G_R$. Now X is normal, that is, $G_R \trianglelefteq A$ if and only if $G_R = \alpha^{-1}G_R\alpha \trianglelefteq \alpha^{-1}A\alpha = A'$. □

By considering the above proposition we have the following result.

Proposition 2.7. *Let G be a finite abelian group and S be a generating set of G not containing the identity 1_G. Assume S satisfies the condition s, t, u, $v \in S$ with*

$$st = uv \neq 1 \Longrightarrow \{s,t\} = \{u,v\}.$$

Then the t-Cayley hypergraph is normal.

Lemma 2.8. *Let $G = G_1 \times G_2$ be the direct product of two finite groups G_1 and G_2, S_1 and S_2 subsets of G_1 and G_2, respectively, and $S = S_1 \cup S_2$ the disjoint union of S_1 and S_2. Then,*
(i) $t\text{-}Cay(G,S) \cong t'\text{-}Cay(G_1,S_1) \times t''\text{-}Cay(G_2,S_2)$, where $t = max\{t', t''\}$.
(ii) If $t\text{-}Cay(G,S)$ is normal, then $t'\text{-}Cay(G_1,S_1)$ is also normal.
(iii) If $t'\text{-}Cay(G_1,S_1)$ and $t''\text{-}Cay(G_2,S_2)$ are both normal and relatively prime, then $t\text{-}Cay(G,S)$ is normal.

Let $X = t\text{-}Cay(G,S)$ be a connected t-Cayley hypergraph of an abelian group G with respect to S, and T the subgroup generated by all non-involutions in S. Set $K = T \cap S$ and $J = S - K$ so that $T = \langle K \rangle$. Let $Y = t\text{-}Cay(T,K)$. If J is independent, then $\langle J \rangle = \mathbb{Z}_2^J$, the direct product of J copies of $\mathbb{Z}_2$. So by Proposition 2.7, $t\text{-}Cay(\langle J \rangle, J)$ is normal for $\langle J \rangle$. From Lemma 2.8, we have the following.

Lemma 2.9. *If $T \cap \langle J \rangle = 1$ and J is independent, then $G = T \times \mathbb{Z}_2^J$ and $X = Y \times t\text{-}Cay(\langle J \rangle, J)$. Moreover, if Y is normal and relatively prime with K_2, then X is normal.*

Proposition 2.10 (4, Proposition 1.10)**.** *A t-Cayley hypergraph $X = t\text{-}Cay(G,S)$ is connected if and only if S is a set of generators for G.*

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