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INEQUALITIES OF THE HERMITE-HADAMARD TYPE, FOR FUNCTIONS (H, M)-CONVEX MODIFIED OF THE SECOND TYPE

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Abstract

In this paper we obtain some extensions of the Hermite-Hadamard Inequality for functions (h, m)-convex modified of second type, using the framework of weighted integrals. We show throughout the work that several known results are particular cases of ours.

Keywords: Hermite-Hadamard integral inequality, integral operators weighted, (h, m)-convex modified functions
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1 Introduction

The notion of convex function has been the object of attention of many researchers in recent years, due to its multiple applications and links with various mathematical areas. Readers interested in the aforementioned development, can consult [24], where a panorama, practically complete, of these branches is presented.

A function $\psi : I \to \mathbb{R}$, $I := [a, b]$ is said to be convex if $\psi\big(\tau\xi + (1-\tau)\varsigma\big) \leq \tau\psi(\xi) + (1-\tau)\psi(\varsigma)$ holds $\forall$ $\xi, \varsigma \in I$,$\tau \in [0, 1]$. And they say that the function ψ is concave on $[a, b]$ if the inequality is the opposite.

In [4] we presented the following definitions.

Definition 1.1. Let $h : [0, 1] \to \mathbb{R}$ be a nonnegative function, $h \neq 0$ and $\psi : I = [0, +\infty) \to [0, +\infty)$. If inequality

$$\psi\left(\tau\xi + m(1-\tau)\varsigma\right) \leq h^s(\tau)\psi(\xi) + m(1 - h^s(\tau))\psi\left(\frac{\varsigma}{m}\right) \tag{1}$$

is fulfilled for all $\xi, \varsigma \in I$ and $\tau \in [0, 1]$, where $m \in [0, 1]$, $s \in [-1, 1]$.Then is said function ψ is a $(h, m)-$convex modified of first type on I.

Definition 1.2. Let $h : [0, 1] \to \mathbb{R}$ nonnegative functions, $h \neq 0$ and $\psi : I = [0, +\infty) \to [0, +\infty)$. If inequality

$$\psi\left(\tau\xi + m(1-\tau)\varsigma\right) \leq h^s(\tau)\psi(\xi) + m(1 - h(\tau))^s\psi\left(\frac{\varsigma}{m}\right) \tag{2}$$

is fulfilled for all $\xi, \varsigma \in I$ and $\tau \in [0, 1]$,where $m \in [0, 1]$, $s \in [-1, 1]$. Then is said function ψ is a $(h, m)-$convex modified of second type on I.

[1]speaker

Remark 1.3. From Definitions 1.1 and 1.2 we can define $N^{s}_{h,m}[a,b]$,where $a,b \in [0,+\infty)$, as the set of functions $(h,m)-$convex modified, for which $\psi(a) \geq 0$, characterized by the triple $(h(\tau), m, s)$. Note that if:

1. $(h(\tau), 0, 0)$ we have the increasing functions ([9]).
2. $(\tau, 0, s)$ we have the $s-$starshaped functions ([9]).
3. $(\tau, 0, 1)$ we have the starshaped functions ([9]).
4. $(\tau, 1, 1)$ then ψ is a convex function on $[0,+\infty)$ ([9]).
5. $(1, 1, s)$ then ψ is a P-convex function on $[0,+\infty)$ ([13]).
6. $(\tau, m, 1)$ then ψ is a $m-$convex function on $[0,+\infty)$ ([33]).
7. $(\tau, 1, s)$ $s \in (0,1]$ then ψ is a $s-$convex function on $[0,+\infty)$ ([8, 18]).
8. $(\tau, 1, s)$ $s \in [-1,1]$ then ψ is a $s-$convex extended function on $[0,+\infty)$ ([34]).
9. (τ, m, s) $s \in (0,1]$ then ψ is a (s,m)-convex extended function on $[0,+\infty)$ ([28]).
10. $(\tau^{a}, 1, s)$ with $a \in (0,1]$, then ψ is a (a,s)-convex function on $[0,+\infty)$ ([7]).
11. $(\tau^{a}, m, 1)$ with $a \in (0,1]$, then ψ is a (a,m)-convex function on $[0,+\infty)$ ([21]).
12. (τ^{a}, m, s) with $a \in (0,1]$, then ψ is a $s-(a,m)$-convex function on $[0,+\infty)$ ([35]).
13. $(h(\tau), m, 1)$ then ψ is a variant of an $(h,m)-$convex function on $[0,+\infty)$ ([27]).

One of the most important inequalities, for convex functions, is the called Hermite–Hadamard inequality:

$$\psi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_{a}^{b}\psi(\xi)d\xi \leq \frac{\psi(a)+\psi(b)}{2} \tag{3}$$

valid for any function ψ convex on the interval $[a,b]$. This inequality was published by Hermite ([16]) in 1883 and, independently, by Hadamard in 1893 ([15]). It gives an estimation of the mean value of a convex function, interpolates the image of the average of the interval and the average of the images of the extremes of the interval and it is important to note that it also provides a refinement to the Jensen inequality. Several results can be consulted in [2, 5, 6, 8, 12, 11, 13, 14, 17, 19, 20, 22, 25, 32] and references therein for more information and other extensions of the Hermite–Hadamard inequality.

All through the work we utilize the functions Γ (see [29, 30, 36, 37]) and Γ_k (see [10]):

$$\Gamma(z) = \int_{0}^{\infty}\tau^{z-1}e^{-\tau}\,\mathrm{d}\tau, \quad \Re(z) > 0,$$
$$\Gamma_k(z) = \int_{0}^{\infty}\tau^{z-1}e^{-\tau^{k}/k}\,\mathrm{d}\tau, k > 0.$$

Unmistakably if $k \to 1$ we have $\Gamma_k(z) \to \Gamma(z)$, $\Gamma_k(z) = (k)^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right)$ and $\Gamma_k(z+k) = z\Gamma_k(z)$.

To encourage comprehension of the subject, we present the definition of Riemann-Liouville fractional integral (with $0 \leq a < \tau < \nu_2 \leq \infty$). The first is the classic Riemann-Liouville fractional integrals.

Definition 1.4. Let $\psi \in L_1[a,b]$. Then the Riemann-Liouville fractional integrals of order $\alpha \in \mathbb{C}$, $\Re(\alpha) > 0$ are defined by (right and left respectively):

$$I^{\alpha}_{a+}\psi(\xi) = \frac{1}{\Gamma(\alpha)}\int_{a}^{\xi}(\xi-\tau)^{\alpha-1}\psi(\tau)\,d\tau, \quad \xi > a$$
$$I^{\alpha}_{b-}\psi(\xi) = \frac{1}{\Gamma(\alpha)}\int_{\xi}^{b}(\tau-\xi)^{\alpha-1}\psi(\tau)\,d\tau, \quad \xi < b.$$

Next we present the weighted integral operators, which will be the basis of our work.

Definition 1.5. Let $\psi \in L_1(a,b)$ and let $w:[0,\infty) \to [0,\infty)$ be a continuous function with first and second order derivatives piecewise continuous on $[0,\infty)$. Then the weighted fractional integrals are defined by (right and left, respectively):

$$
\begin{aligned}
{}^{n+1}J^{w}_{a+}\psi(\xi) &= \int_a^{\xi} w''\left(\frac{\xi-\tau}{\frac{b-a}{n+1}}\right)\psi(\tau)\,d\tau, \quad \xi > a,\\
{}^{n+1}J^{w}_{b-}\psi(\xi) &= \int_{\xi}^{b} w''\left(\frac{\tau-\xi}{\frac{b-a}{n+1}}\right)\psi(\tau)\,d\tau, \quad \xi < b.
\end{aligned}
$$

Remark 1.6. To have a clearer idea of the amplitude of the Definition 1.5, let's consider some particular cases:

1. Putting $w''(\tau) \equiv 1$, $n = 0$, we obtain the classical Riemann integral.
2. If $w''(\tau) = \frac{\tau^{(\alpha-1)}(b-a)^{(\alpha-1)}}{\Gamma(\alpha)}$ for $n = 0$, then we obtain the Riemann-Liouville fractional integral, right and left.
3. If $w''(\tau) = \frac{\tau^{(\frac{\alpha}{k}-1)}(\nu_2-a)^{(\alpha-1)}}{k\Gamma_k(\alpha)}$ for $n = 0$, then we obtain the k-Riemann-Liouville fractional integral, right and left ([23]).
4. If $w''(\xi-\tau) = \frac{(h(\xi)-h(\tau))^{(\frac{\alpha}{k}-1)}h'(\tau)(b-a)^{(\alpha-1)}}{k\Gamma_k(\alpha)}$ and $w''(\tau-\xi) = \frac{(h(\tau)-h(\xi))^{(\frac{\alpha}{k}-1)}h'(\tau)(b-a)^{(\alpha-1)}}{k\Gamma_k(\alpha)}$, $n = 0$, then we obtain the right and left-sided fractional integrals of a function ψ with respect to another function h on $[a,b]$ (see [1]).

In this paper, we obtain several integral inequalities of the Hermite-Hadamard type for twice differentiable (h,m)-convex second-order modified functions. To obtain the inequalities, we use weighted integral operators, previously defined.

2 Hermite-Hadamard type inequalities for (h,m)-convex modified functions of second type

Below, we provide a result that will be useful throughout the work.

Lemma 2.1. *Let ψ be a real function defined on some interval $[a,b] \subset \mathbb{R}$, differentiable on (a,b). If $\psi' \in L_1(a,b)$, then we have the following equality:*

$$
\begin{aligned}
&\frac{n+1}{b-a}\left\{w(0)\left[\psi'(b)+\psi'(a)\right]-w(1)\left(\psi'(\frac{an+b}{n+1})+\psi'(\frac{bn+a}{n+1})\right)\right\} \\
&+\left(\frac{n+1}{b-a}\right)\left\{w'(0)\left(\psi(a)-\psi(b)\right)-w'(1)\left[\psi\left(\frac{an+b}{n+1}\right)+\psi\left(\frac{bn+a}{n+1}\right)\right]\right\} \\
&-\left(\frac{n+1}{b-a}\right)^3\left({}^{(n+1)}J^{w}_{b-}\psi\left(\frac{bn+a}{n+1}\right)+{}^{(n+1)}J^{w}_{a+}\psi\left(\frac{an+b}{n+1}\right)\right) \\
=&\int_0^1 w(\tau)\left[\psi''\left(\frac{\tau}{n+1}a+\frac{n+1-\tau}{n+1}b\right)+\psi''\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)\right]d\tau.
\end{aligned}
\tag{4}
$$

Proof. First note that

$$\begin{aligned}
&\int_0^1 w(\tau)\left[\psi''\left(\frac{\tau}{n+1}a+\frac{n+1-\tau}{n+1}b\right)+\psi''\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)\right]d\tau\\
&=\int_0^1 w(\tau)\psi''\left(\frac{\tau}{n+1}a+\frac{n+1-\tau}{n+1}b\right)d\tau\\
&+\int_0^1 w(\tau)\psi''\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)d\tau\\
&= I_1+ I_2.
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
I_1 &= \left(\frac{n+1}{b-a}\right)\left[w(1)\psi'(b)-w(0)\psi'\left(\frac{bn+a}{n+1}\right)\right]-\left(\frac{n+1}{b-a}\right)^2\left[w'(1)\psi(b)-w'(0)\psi\left(\frac{bn+a}{n+1}\right)\right]\\
&+ \left(\frac{n+1}{b-a}\right)^3 {}^{(n+1)}J^w_{b-}\psi(\left(\frac{bn+a}{n+1}\right))
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= -\left(\frac{n+1}{b-a}\right)\left[w(1)\psi'\left(\frac{an+b}{n+1}\right)-w(0)\psi'(b)\right]+\left(\frac{n+1}{b-a}\right)^2\left[w'(1)\psi\left(\frac{an+b}{n+1}\right)-w'(0)(a)\psi\right]\\
&+ \left(\frac{n+1}{b-a}\right)^2\int_0^1 w''(\tau)\psi\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)d\tau\\
&= -\left(\frac{n+1}{b-a}\right)\left[w(1)\psi'\left(\frac{an+b}{n+1}\right)-w(0)\psi'(b)\right]+\left(\frac{n+1}{b-a}\right)^2\left[w'(1)\psi\left(\frac{an+b}{n+1}\right)-w'(0)(a)\psi\right]\\
&- \left(\frac{n+1}{b-a}\right)^3 {}^{(n+1)}J^w_{a+}\psi\left(\frac{an+b}{n+1}\right),
\end{aligned}$$

since

$$\int_0^1 w''(\tau)\psi\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)d\tau=-\left(\frac{n+1}{b-a}\right)\int_a^{\left(\frac{an+b}{n+1}\right)} w''\left(\frac{\xi-a}{\frac{b-a}{n+1}}\right)\psi(\xi)d\xi.$$

From I_1+I_2, and grouping appropriately, we have the required inequality. □

Remark 2.2. To indicate two particular cases of the previous result, if we consider $n=0$ and $w(t)=t(1-t)$ we obtain the Lemma 1 of [3]; and putting $n=1$ and $w(t)=[tB_t(n+1,\alpha-n)-B_t(n+2,\alpha-n)]$ where $\alpha\in(n,n+1]$, with B_t the beta function, we have the Lemma 2.1 of [31].

Our first main result is the following.

Theorem 2.3. *Let $\psi: I\subset\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable function on I° such that $\psi''\in L_1\left[a,\frac{b}{m}\right]$. Under the assumptions of Lemma 2.1 if $|\psi''|$ is modified (h,m)-convex of second type on $\left[a,\frac{b}{m}\right]$, we have the following inequality:*

$$\begin{aligned}
&\left|\mathbb{A}+\left({}^{(n+1)}J^w_{b-}\psi\left(\frac{bn+a}{n+1}\right)+{}^{(n+1)}J^w_{a+}\psi\left(\frac{an+b}{n+1}\right)\right)\right| \\
&\leq\left(\frac{b-a}{n+1}\right)^3\left\{\left(\left|\psi''(a)\right|+\left|\psi''(b)\right|\mathbb{B}+m\mathbb{C}\left[\left|\psi''\left(\frac{a}{m}\right)\right|+\left|\psi''\left(\frac{b}{m}\right)\right|\right]\right)\right\}
\end{aligned} \tag{5}$$

with

$$\begin{aligned}\mathbb{A} &= \left(\frac{b-a}{n+1}\right)^2 \left\{w(0)\left[\psi'(b)+\psi'(a)\right] - w(1)\left(\psi'(\frac{an+b}{n+1})+\psi'(\frac{bn+a}{n+1})\right)\right\} \\ &+ \left(\frac{b-a}{n+1}\right)\left\{w'(0)\left(\psi(a)-\psi(b)\right) - w'(1)\left[\psi\left(\frac{an+b}{n+1}\right)+\psi\left(\frac{bn+a}{n+1}\right)\right]\right\},\end{aligned}$$

$\mathbb{B} = \int_0^1 w(\tau)h^s\left(\frac{\tau}{n+1}\right)d\tau$, *and* $\mathbb{C} = \int_0^1 w(\tau)\left(1-h\left(\frac{n+1-\tau}{n+1}\right)\right)^s d\tau$.

Proof. From Lemma 2.1 we obtain

$$\begin{aligned}&\left|\int_0^1 w(\tau)\left[\psi''\left(\frac{\tau}{n+1}a+\frac{n+1-\tau}{n+1}b\right)+\psi''\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)\right]d\tau\right| \\ \leq &\int_0^1 w(\tau)\left|\psi''\left(\frac{\tau}{n+1}a+\frac{n+1-\tau}{n+1}b\right)\right|d\tau \\ + &\int_0^1 w(\tau)\left|\psi''\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)\right|d\tau.\end{aligned}$$

Using the modified (h,m)-convexity of $|\psi''|$, we get

$$\begin{aligned}&\int_0^1 w(\tau)\left|\psi''\left(\frac{\tau}{n+1}a+\frac{n+1-\tau}{n+1}b\right)\right|d\tau \\ &\leq \int_0^1 w(\tau)\left[h^s\left(\frac{\tau}{n+1}\right)\left|\psi''(a)\right| + m\left(1-h\left(\frac{n+1-\tau}{n+1}\right)\right)^s\left|\psi''\left(\frac{b}{m}\right)\right|\right]d\tau \\ &= \left|\psi''(a)\right|\int_0^1 w(\tau)h^s\left(\frac{\tau}{n+1}\right)d\tau + m\left|\psi''\left(\frac{b}{m}\right)\right|\int_0^1 w(\tau)\left(1-h\left(\frac{n+1-\tau}{n+1}\right)\right)^s d\tau.\end{aligned} \tag{6}$$

In the same way

$$\begin{aligned}&\int_0^1 w(\tau)\left|\psi''\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)\right|d\tau \\ &\leq \left|\psi''(b)\right|\int_0^1 w(\tau)h^s\left(\frac{\tau}{n+1}\right)d\tau + m\left|\psi''\left(\frac{a}{m}\right)\right|\int_0^1 w(\tau)\left(1-h\left(\frac{n+1-\tau}{n+1}\right)\right)^s d\tau.\end{aligned} \tag{7}$$

From (6) and (7) we easily obtain (5). In this way the theorem is proved. □

Remark 2.4. With $h(t) = t^a$ and $w(t) = (1-t)^{1+\alpha}$ from the above result we obtain Theorem 2.1 of [26].

Refinements of the previous results, can be obtained by imposing new additional conditions on $|\psi''|^q$.

Theorem 2.5. *Let* $\psi : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ *be a differentiable function on* I° *such that* $\psi'' \in L_1\left[a, \frac{b}{m}\right]$. *Under the assumptions of Lemma 2.1 if* $|\psi''|^q$, $q \geq 1$, *is modified* $(h,m)-$*convex of second type on* $\left[a, \frac{b}{m}\right]$, *we have the following inequality:*

$$\begin{aligned}&\left|\mathbb{A} + \left({}^{(n+1)}J^w_{b-}\psi\left(\frac{bn+a}{n+1}\right) + {}^{(n+1)}J^w_{a+}\psi\left(\frac{an+b}{n+1}\right)\right)\right| \\ &\leq \left(\frac{b-a}{n+1}\right)^3 B_q\left\{\left(p_{11}C_{11}+mp_2C_{12}\right)^{\frac{1}{q}} + \left(p_{22}C_{11}+mp_1C_{12}\right)^{\frac{1}{q}}\right\}\end{aligned} \tag{8}$$

with $\mathbb{A}$ *as before,* $B_q = \left(\int_0^1 w^p(\tau)\,d\tau\right)^{\frac{1}{p}}$, $p_1 = \left|\psi''\left(\frac{a}{m}\right)\right|^q$, $p_{11} = |\psi''(a)|^q$, $p_{22} = |\psi''(b)|^q$, $p_2 = \left|\psi''\left(\frac{b}{m}\right)\right|^q$, $C_{11} = \int_0^1 h^s\left(\frac{\tau}{n+1}\right)d\tau$, *and* $C_{12} = \int_0^1\left(1-h\left(\frac{n+1-\tau}{n+1}\right)\right)^s d\tau$.

Proof. As previous result, from Lemma 2.1 we obtain

$$\begin{aligned}
&\left|\int_0^1 w(\tau)\left[\psi''\left(\frac{\tau}{n+1}a+\frac{n+1-\tau}{n+1}b\right)+\psi''\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)\right]d\tau\right| \\
\leq\ & \int_0^1 w(\tau)\left|\psi''\left(\frac{\tau}{n+1}a+\frac{n+1-\tau}{n+1}b\right)\right|d\tau \\
+\ & \int_0^1 w(\tau)\left|\psi''\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)\right|d\tau.
\end{aligned}$$

From Hölder's inequality, we obtain

$$\begin{aligned}
&\int_0^1 w(\tau)\left|\psi''\left(\frac{\tau}{n+1}a+\frac{n+1-\tau}{n+1}b\right)\right|d\tau \\
\leq\ & \left(\int_0^1 w^p(\tau)d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi''\left(\frac{\tau}{n+1}a+\frac{n+1-\tau}{n+1}b\right)\right|^q d\tau\right)^{\frac{1}{q}}
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
&\int_0^1 w(\tau)\left|\psi''\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)\right|d\tau \\
\leq\ & \left(\int_0^1 w^p(\tau)d\tau\right)^{\frac{1}{p}}\left(\int_0^1\left|\psi''\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)\right|^q d\tau\right)^{\frac{1}{q}}
\end{aligned} \tag{10}$$

for $\frac{1}{p}+\frac{1}{q}=1$. Using the (h,m)-convexity of the second type of $|\psi''|^q$, we obtain from (9) and (10):

$$\begin{aligned}
&\int_0^1\left|\psi''\left(\frac{\tau}{n+1}a+\frac{n+1-\tau}{n+1}b\right)\right|^q d\tau \\
\leq\ & \left|\psi''(a)\right|^q\int_0^1 h^s\left(\frac{\tau}{n+1}\right)d\tau+m\left|\psi''\left(\frac{b}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{n+1-\tau}{n+1}\right)\right)^s d\tau,
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\int_0^1\left|\psi''\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)\right|^q d\tau \\
\leq\ & \left|\psi''(b)\right|^q\int_0^1 h^s\left(\frac{\tau}{n+1}\right)d\tau+m\left|\psi''\left(\frac{a}{m}\right)\right|^q\int_0^1\left(1-h\left(\frac{n+1-\tau}{n+1}\right)\right)^s d\tau.
\end{aligned} \tag{12}$$

Denoting, for brevity $B_q=\left(\int_0^1 w^p(\tau)\,d\tau\right)^{\frac{1}{p}}$, substituting (11), (12) in (9) and (10), we obtain the required inequality. □

Remark 2.6. If we consider s-convex functions of the second type, this Theorem contains Theorem 2.3 of [26].

Theorem 2.7. *Let $\psi: I\subset\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable function on I° such that $\psi'\in L_1\left[a,\frac{b}{m}\right]$. Under the assumptions of Lemma 2.1 if $|\psi'|^q$, $q>1$, is modified (h,m)-convex of second type on $\left[a,\frac{b}{m}\right]$, we have the following inequality:*

$$\begin{aligned}
&\left|\mathbb{A}+\left({}^{(n+1)}J^w_{b-}\psi\left(\frac{bn+a}{n+1}\right)+{}^{(n+1)}J^w_{a+}\psi\left(\frac{an+b}{n+1}\right)\right)\right| \\
&\leq\left(\frac{b-a}{n+1}\right)^3\Delta\left\{\left(p_{11}D_{11}+mp_2D_{12}\right)^{\frac{1}{q}}+\left(p_{22}D_{21}+mp_2D_{22}\right)^{\frac{1}{q}}\right\}
\end{aligned} \tag{13}$$

with $\mathbb{A}$, p_1, p_{11}, p_{22} *and* p_2 *as before,* $\Delta = \left(\int_0^1 w(\tau)\,d\tau\right)^{1-\frac{1}{q}}$, $D_{11} = \int_0^1 w(\tau)h^s\left(\frac{\tau}{n+1}\right)d\tau$, $D_{12} = \int_0^1 w(\tau)\left(1-h\left(\frac{n+1-\tau}{n+1}\right)\right)^s d\tau$, $D_{21} = \int_0^1 w(\tau)h^s\left(\frac{n+1-\tau}{n+1}\right)d\tau$ *and* $D_{22} = \int_0^1 w(\tau)\left(1-h\left(\frac{\tau}{n+1}\right)\right)^s d\tau$.

Proof. As before, from the Lemma 2.1 we have:

$$\begin{aligned}
&\left|\int_0^1 w(\tau)\left[\psi''\left(\frac{\tau}{n+1}a+\frac{n+1-\tau}{n+1}b\right)+\psi''\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)\right]d\tau\right| \\
\leq\ & \int_0^1 w(\tau)\left|\psi''\left(\frac{\tau}{n+1}a+\frac{n+1-\tau}{n+1}b\right)\right|d\tau \\
+\ & \int_0^1 w(\tau)\left|\psi''\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)\right|d\tau.
\end{aligned}$$

and using well known power mean inequality, we have

$$\begin{aligned}
&\int_0^1 w(\tau)\left|\psi''\left(\frac{\tau}{n+1}a+\frac{n+1-\tau}{n+1}b\right)\right|d\tau \\
\leq\ & \left(\int_0^1 w(\tau)d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\psi''\left(\frac{\tau}{n+1}a+\frac{n+1-\tau}{n+1}b\right)\right|^q d\tau\right)^{\frac{1}{q}}
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&\int_0^1 w(\tau)\left|\psi''\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)\right|d\tau \\
\leq\ & \left(\int_0^1 w(\tau)d\tau\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\psi''\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)\right|^q d\tau\right)^{\frac{1}{q}}
\end{aligned} \tag{15}$$

Using the modified (h,m)-convexity of $|\psi'|^q$, we get

$$\begin{aligned}
&\int_0^1\left|\psi''\left(\frac{\tau}{n+1}a+\frac{n+1-\tau}{n+1}b\right)\right|^q d\tau \\
&\leq \int_0^1 w(\tau)\left[h^s\left(\frac{\tau}{n+1}\right)\left|\psi''(a)\right|^q+m\left(1-h\left(\frac{n+1-\tau}{n+1}\right)\right)^s\left|\psi''(\frac{b}{m})\right|^q\right]d\tau \\
&= \left|\psi''(a)\right|^q\int_0^1 w(\tau)h^s\left(\frac{\tau}{n+1}\right)d\tau+m\left|\psi''(\frac{b}{m})\right|^q\int_0^1 w(\tau)\left(1-h\left(\frac{n+1-\tau}{n+1}\right)\right)^s d\tau.
\end{aligned} \tag{16}$$

Similarly

$$\begin{aligned}
&\int_0^1 w(\tau)\left|\psi''\left(\frac{\tau}{n+1}b+\frac{n+1-\tau}{n+1}a\right)\right|d\tau \\
&\leq \left|\psi''(b)\right|^q\int_0^1 w(\tau)h^s\left(\frac{\tau}{n+1}\right)d\tau+m\left|\psi''(\frac{b}{m})\right|^q\int_0^1 w(\tau)\left(1-h\left(\frac{n+1-\tau}{n+1}\right)\right)^s d\tau.
\end{aligned} \tag{17}$$

If we put (16) and (17), in (14) and in (15), it allows us to obtain the inequality (13). In this way the proof is completed. □

Remark 2.8. Theorems 2.2 and 2.4 of [26], referring to s-convex functions of the second type, are particular cases of the previous result.

3 Conclusions

This work covers a fairly extensive panorama of Hermite-Hadamard inequalities for functions whose second derivatives are (h, m)-convex modified of the second type, by means of weieghted integrals. Both the definition of convexity used, as well as the use of general integral operators (for appropriate kernel choices we can obtain the classical integral operator, fractional integrals of the Riemann-Liouville type or generalized integrals not necessarily fractional) and the general notation to represent the argument of function, allow us to generalize many results known from the literature. It remains to be obtained, similar results for convex functions (h, m) of the first type.

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