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The wavelet basis approach to study of first kind integral equation with degenerate kernel

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Abstract

In this article we study about Shannon wavelet in $\mathcal{L}_\alpha$ space for every positive α, then by using this approximation for ill-posed Fredholm integral equation of the first kind and using collocation method, we try to estimate the solution of integral equation. The exponential convergence rate of the method $O(\exp(-cN^{1/2}))$ is proved. Finally, convergence of this method is discussed.

Keywords: Integral equation of the first kind, Collocation method, Shannon wavelet, Sinc approximation.

Mathematics Subject Classification [2010]: 45B05, 65L60, 47A05, 65D30

1 Introduction

Fredholm integral equation of the first kind is known as

$$\int_a^b k(s,t)f(t)dt = g(s), \qquad -\infty < a \leq s \leq b < \infty, \tag{1}$$

where $k(s,t)$ and $g(s)$ are known functions, but $f(t)$ is an unknown function that we try to estimate it. Every integral operator with degenerate kernel has $Rang(\mathcal{K})$ with finite dimension. So that integral operator with degenerate kernel has a bounded inverse [1].
In many cases, we can not solve this equation analytically, although for proving existence and uniqueness of solution of integral equation has been done lots of researches [2–4], but analytical solutions for those often are not available. So that by using numerical methods we try to estimate a solution for this equation. In this paper we use collocation method for solving it. In collocation method we need some basis functions to approximate the solution of this equation. In this work we choose Sinc function as a base function and use some theorems for introducing an approximation for solution of integral equation of the first kind.
Two kinds of Shannon wavelets can be implemented: real Shannon wavelet and complex Shannon wavelet. The signal analysis by ideal pass-band filters define a decomposition known as Shannon wavelets (or sinc wavelets). The Haar and sinc systems are Fourier duals of each other.
Real Shannon wavelet: The spectrum of the Shannon mother wavelet is given by:

$$\Psi^{(\mathrm{Sha})}(w) = \Pi(\frac{w-3\pi/2}{\pi}) + \Pi(\frac{w+3\pi/2}{\pi}),$$

[1]speaker

where the (normalised) gate function is defined by

$$\Pi(x) = \begin{cases} 1, & \text{if } |x| \leq 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

The analytical expression of the real Shannon wavelet can be found by taking the inverse Fourier transform:

$$\psi^{(\mathrm{Sha})}(t) = \mathrm{Sa}\left(\frac{\pi t}{2}\right) \cdot \cos\left(\frac{3\pi t}{2}\right),$$

or alternatively as

$$\psi^{(Sha)}(t) = 2 \cdot \mathrm{sinc}(2t) - \mathrm{sinc}(t),$$

where

$$\mathrm{sinc}(t) = \frac{\sin \pi t}{\pi t},$$

is the usual sinc function that appears in Shannon sampling theorem. This wavelet belongs to C^{∞}- class, but it decreases slowly at infinity and has no bounded support, since band-limited signals cannot be time-limited. The scaling function for the Shannon MRA (or Sinc-MRA) is given by the sample function:

$$\phi^{(Sha)}(t) = \frac{\sin \pi t}{\pi t} = \mathrm{sinc}(t).$$

In the case of complex continuous wavelet, the Shannon wavelet is defined by

$$\psi^{(CSha)}(t) = sinc(t).e^{-j2\pi t}.$$

2 Sinc Approximation

In this section we introduce the cardinal function and some of its properties. For this result $sinc(x)$ definition is followed by

$$sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Now, for $h > 0$ and integer k, we define k'th Sinc function with step size h by [5],

$$S(k,h)(x) = \frac{\sin(\pi(x-kh)/h)}{\pi(x-kh)/h}.$$

Definition 2.1. Let $h > 0$, and let $W(\frac{\pi}{h})$ denote the family of all functions f that are analytic in $\mathcal{C}$, such that

$$\int_{\mathbb{R}} |f(t)|^2 dt < \infty,$$

and for all $z \in \mathcal{C}$, $|f(z)| \leq c \exp(\pi|z|/h)$ with c a positive constant.

Theorem 2.2. *([6]) Let $h > 0$, the sequence $\{h^{-1/2}S(k,h)\}_{k=-\infty}^{\infty}$ is a complete orthonormal sequence in $W(\frac{\pi}{h})$. Every f in $W(\frac{\pi}{h})$ has cardinal series representation $f(x) = C(f,h)(x)$ with*

$$C(f,h) = \sum_{k=-\infty}^{\infty} f(kh)S(k,h), \tag{2}$$

moreover;

$$\lim_{N\to\infty} f(x)dx = \lim_{N\to\infty} h \sum_{k=-N}^{N} f(kh).$$

Theorem 2.3. *([6]) Let $f \in \mathcal{L}_{\alpha}(D_d)$ for $\alpha > 0$ and taking $h = (\frac{\pi d}{\alpha N})^{1/2}$ then there exists a positive number c_1, depending only on f, d, α, y such that for $s = 2$ or $s = \infty$*

$$\begin{aligned} \|E_N(f,h)(.+iy)\|_s &= \|f(.+iy) - \sum_{k=-N}^{N} f(kh)S(k,h)(.+iy)\|_s \\ &\leq c_1 N^{(1-1/s)/2} \exp\{-(\frac{\pi\alpha}{d})^{1/2}(d-|y|)N^{1/2}\}. \end{aligned} \tag{3}$$

3 Collocation Method

In this section we introduce a numerical method to solve Eq. (1). For this result, we should choose a family of functions with finite dimension, then estimate the exact solution by them. Methods that use this strategy are called projection methods. One of these methods is collocation method.

In this paper we assume X_n is a subspace of $\mathcal{L}_\alpha$ which is generated by sequence of orthonormal basis functions $\{h^{1/2}S(k,h)(x)\}_{k=-N}^{N}$. So that we can approximate unknown function $f(t)$ by them. So we have

$$f(t) \approx P_N(f(t)) = \sum_{k=-N}^{N} f(kh)S(k,h)(t), \qquad \forall h > 0, \tag{4}$$

now by substituting this in integral equation we get

$$\int_a^b k(s,t) \sum_{k=-N}^{N} f(kh)S(k,h)(t)dt = g(s),$$

so we define residual equation as

$$R_N(s) = \sum_{k=-N}^{N} f(kh)S(k,h)(s) - \int_a^b k(s,t) \sum_{k=-N}^{N} f(kh)S(k,h)(t)dt - g(s), \tag{5}$$

for determining the unknown coefficients $f(kh)$'s we select some collocation points such that

$$R_N(s_i) = 0, \qquad i = 0, ..., 2N,$$

in this paper collocation points are

$$s_i = a + \frac{i(b-a)}{2N}, \qquad i = 0, ..., 2N,$$

so that we have a system of linear equations $A_N X = b_N$ where

$$\begin{aligned} A_N &= [\int_a^b k(s_i,t)S(k,h)(t)dt]_{k=-N}^{N}, \quad i = 0,1,...,2N, \\ X^T &= [f(kh)]_{k=-N}^{N}, \\ b_N &= [g(s_i)] \quad i = 0,1,...,2N. \end{aligned}$$

4 Convergence Study

In this section we discuss about convergence of Sinc approximation for Fredholm integral equation of the first kind. For this result we have the following theorem.

Theorem 4.1. *([7]) Assume that integral operator $\mathcal{K} : X \to Y$ as defined by Eq. (4) is a compact linear operator with degenerate kernel and this operator is one to one and onto. Also assume that $\|\mathcal{K} - P_N\mathcal{K}\| \to 0; \quad N \to \infty$ with maximum norm where P_N is a projection operator that was defined by Eq. (8). Then $P_N\mathcal{K}$ has a bounded inverse moreover*

$$\frac{\|f - f_N\|}{\|f\|} \le \|(P_N\mathcal{K})^{-1}\|.\|P_N\|.\|\mathcal{K} - P_N\mathcal{K}\|. \tag{6}$$

Lemma 4.2. *If $\mathcal{K}$ is an integral operator defined by Eq. (4) with degenerate kernel and P_N is a projection operator defined by Eq. (8) then*

$$\|\mathcal{K} - P_N\mathcal{K}\| \to 0, \quad N \to \infty.$$

Proof.

$$\begin{aligned}\|\mathcal{K}-P_N\mathcal{K}\| &= \sup_{s\in[a,b]}|\mathcal{K}(f(t))-P_N\mathcal{K}(f(t))|\\ &= \sup_{s\in[a,b]}|\int_a^b(k(s,t)-\sum_{j=-N}^{N}k(jh,t)S(j,h)(s))f(t)dt|,\end{aligned}$$

now we can write

$$\begin{aligned}\|\mathcal{K}-P_N\mathcal{K}\| &= \sup_{s\in[a,b]}|\int_a^b(\sum_{i=1}^{m}a_i(s)b_i(t)-\sum_{j=-N}^{N}\sum_{i=1}^{m}a_i(jh)b_i(t)S(j,h)(s))f(t)dt|\\ &\leq \sup_{s\in[a,b]}|\sum_{i=1}^{m}((a_i(s)-\sum_{j=-N}^{N}a_i(jh)S(j,h)(s))(\int_a^b b_i^2(t)dt)^{1/2}(\int_a^b f^2(t)dt)^{1/2})|,\end{aligned}$$

by using Theorem 2.3 we have

$$\sup_{s\in[a,b]}|(a_i(s)-\sum_{j=-N}^{N}a_i(jh)S(j,h)(s))|\leq c_iN^{1/2}exp\{-c_2N^{1/2}\},$$

if we assume $C=Max\{c_i\}$ for $i=1,...,m$ then

$$\sup_{s\in[a,b]}|(\sum_{i=1}^{m}(a_i(s)-\sum_{j=-N}^{N}a_i(jh)S(j,h)(s)))|\leq mCN^{1/2}exp\{-c_2N^{1/2}\},$$

also by regard to this fact that $f(t),b_i(t)\in\mathcal{L}_\alpha\subseteq\mathcal{L}_2$ so that

$$\begin{aligned}M_1 &= Max\{(\int_a^b b_i^2(t)dt)^{1/2}\}<\infty, \quad i=1,...,m,\\ M_2 &= (\int_a^b f^2(t)dt)^{1/2}<\infty,\end{aligned}$$

finally, we have

$$\|\mathcal{K}-P_N\mathcal{K}\|\leq mCN^{1/2}exp\{-c_2N^{1/2}\}M_1M_2\to 0, \quad N\to\infty. \tag{7}$$

□

Now by using Lemma 4.2 and Theorem 4.1 we conclude

$$\frac{\|f-f_N\|}{\|f\|}\to 0, \quad N\to\infty,$$

where f is the exact solution of integral equation and f_N is the Sinc approximation of f defined by Eq. (8). In the following, we want to study on convergence of projection method with Sinc approximation on the ill-posed integral equation of the first kind without any restriction on kernel on integral equation.

Theorem 4.3. *Consider Fredholm integral Eq.* (1)*, we assume that $k(s,t)$ is continuous on square $[a,b]^2$ and $f(t)\in\mathcal{L}_\alpha([a,b])$ for $\alpha>0$, also for $h>0$; $P_N^{num}(f)(t)=\sum_{k=-N}^{N}f^{num}(kh)S(k,h)(t)$ is Sinc approximation of f. Now by using collocation method we have system of equation $A_NX=b_N$. If A_N is nonsingular then*

$$\|f-P_N^{num}(f)\|_\infty\leq[c_1+c_3\|A_N^{-1}\|_\infty N]N^{1/2}\exp\{-c_2N^{1/2}\}.$$

Proof. Let $f(t) \simeq P_N^{num}(f)(t) = \sum_{k=-N}^{N} f^{num}(kh)S(k,h)(t)$ where $f^{num}(kh)$ are unknown coefficients which are found by solving linear system of equations, but we have an other approximation of $f(t)$ which is defined by

$$P_N(f)(t) = \sum_{k=-N}^{N} f(kh)S(k,h)(t).$$

If we substitute $P_N^{num}(f)(t)$ as an approximation of $f(t)$ in Eq. (1) then

$$g(s) = \int_a^b k(s,t)P_N^{num}(f)(t)dt, \tag{8}$$

but if we use $P_N(f)(t)$ then we have

$$\hat{g}(s) = \int_a^b k(s,t)P_N(f)(t)dt. \tag{9}$$

Now, if we convert Eq. (9) to linear system then by solving this system we have

$$[f^{num}(kh)]_{k=-N}^{N} = A_N^{-1}[g(s_i)]_{i=-N}^{N},$$

but, if we convert Eq. (10) to linear system of equations then by solving it we have

$$[f(kh)]_{k=-N}^{N} = A_N^{-1}[\hat{g}(s_i)]_{i=-N}^{N}.$$

So that

$$\sup_{k\in S_N} |f^{num}(kh) - f(kh)| \leq \|A_N^{-1}\| \sup_{i\in S_N} |g(s_i) - \hat{g}(s_i)|, \tag{10}$$

where S_N is all integers belong to $[-N, N]$. Now we should define $\hat{g}(s)$, so

$$\int_a^b k(s,t)P_N(f)(t)dt = g(s) - \int_a^b k(s,t)[f(t) - P_N(f)(t)]dt,$$

then we can assume

$$\hat{g}(s) = g(s) - \int_a^b k(s,t)[f(t) - P_N(f)(t)]dt,$$

so that we have

$$\begin{aligned} \sup_{i\in S_N} |\hat{g}(s_i) - g(s_i)| &= \sup_{s\in[a,b]} \left|\int_a^b k(s,t)[f(t) - P_N(f)(t)]dt\right| \\ &\leq (b-a) \sup_{t,s\in[a,b]} |k(s,t)| \|f - P_N(f)\|, \end{aligned}$$

because $k(s,t)$ is continuous on $[a,b]^2$ so that $M = \sup_{t,s\in[a,b]} |k(s,t)|$ also we have $\|f - P_N(f)\| \leq c_1 N^{1/2} \exp\{-c_2 N^{1/2}\}$ then

$$\sup_{i\in S_N} |g(s_i) - \hat{g}(s_i)| \leq (b-a)Mc_1 N^{1/2} \exp\{-c_2 N^{1/2}\}. \tag{11}$$

Finally, by substituting equation (7) in Eq. (11) we have

$$\sup_{k\in S_N} |f^{num}(kh) - f(kh)| \leq c_3 \|A_N^{-1}\| N^{1/2} \exp\{-c_2 N^{1/2}\}.$$

Also we need to determine a bound for $\|P_N(f)(t) - P_N^{num}(f)(t)\|$ hence

$$\begin{aligned} \sup_{t\in[a,b]} |P_N(f)(t) - P_N^{num}(f)(t)| &= \sup_{t\in[a,b]} \left|\sum_{k=-N}^{N} [f(kh) - f^{num}(kh)]S(k,h)(t)\right| \\ &\leq \|A_N^{-1}\| c_3 N^{1/2} \exp\{-c_2 N^{1/2}\} \sup_{t\in[a,b]} \sum_{k=-N}^{N} |S(k,h)(t)|, \end{aligned}$$

Also [6]

$$\sup_{t\in[a,b]} \sum_{k=-N}^{N} |S(k,h)(t)| \leq \frac{2}{\pi}\{3+\log(N)\},$$

for each sufficiently large N we can replace N instead of $\frac{2}{\pi}\{3+\log(N)\}$ so that

$$\sup_{t\in[a,b]} |P_N(f)(t) - P_N^{num}(f)(t)| \leq c_3\|A_N^{-1}\|N^{3/2}\exp\{-c_2N^{1/2}\},$$

finally,

$$\begin{aligned} \|f(t) - P_N^{num}(f)(t)\| &\leq c_1N^{1/2}\exp\{-c_2N^{1/2}\} + c_3\|A_N^{-1}\|N^{3/2}\exp\{-c_2N^{1/2}\} \\ &= N^{1/2}\exp\{-c_2N^{1/2}\}[c_1 + c_3\|A_N^{-1}\|N], \end{aligned} \tag{12}$$

so that proof of this theorem is completed. □

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