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Seidel Energy of k-fold and Strong k-fold Graphs

Harishchandra S. Ramane[1]
Department of Mathematics, Karnatak University, Dharwad - 580003, India

B. Parvathalu
Department of Mathematics, Karnatak University's Karnatak Arts College, Dharwad - 580001, India

K. Ashoka
Department of Mathematics, Karnatak University Dharwad - 580003, India

Abstract

The Seidel energy of a graph is the sum of absolute values of the eigenvalues of its Seidel matrix. In this paper, an explicit expression for the Seidel energy of k-fold graphs and strong k-fold graphs is obtained. As a consequence, certain Seidel equienergetic graphs are characterized. Moreover, some new class of Seidel equienergetic graphs are presented.

Keywords: Seidel energy, Double graph, k-fold graph, Strong double graph, Strong k-fold graph
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1 Introduction

The most elaborated matrix corresponding to a graph G with n vertices is the *adjacency matrix* $A(G) = [a_{ij}]$, defined by $a_{ij} = 1$ if a vertex v_i is adjacent to a vertex v_j and 0 otherwise. Another well known matrix corresponding to a graph is the *Seidel matrix* $S(G)$ [20] introduced by van Lint and Seidel in 1966. It is defined as $S(G) = J_n - I - 2A(G)$, where J_n is the matrix with all its entries equal to 1 and I is an identity matrix both of same order $n \times n$. The one of important spectral properties of Seidel matrix is that the multiplicity of least Seidel eigenvalue has a connection with equiangular lines in Euclidean space [3]. The *energy* of a graph G is the sum of absolute values of the eigenvalues of G [5]. Haemers introduced the *Seidel energy* [6] of a graph G, defined as sum of absolute values of the Seidel eigenvalues of G and shown a connection with the energy of G. The study on Seidel energy of a graph can be found in [1, 2, 11, 16, 19]. In the study of Seidel energy of a graph, finding the class of graphs with different Seidel eigenvalues which have same Seidel energy is an interesting direction. In this paper, we find the Seidel energy of k-fold graph and strong k-fold graph in terms of Seidel energy of original graph together with some other graph parameters. As a result we characterize some class of graphs with same Seidel energy.

2 Preliminaries

All the graphs in this paper are simple and undirected. Let the $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of a graph G with n vertices $v_1, v_2, \ldots, v_n$. The *degree* d_i of a vertex v_i is the number of edges which are incident with v_i. A graph G is said to be r*-regular* if $d_i = r$ to each vertex $v_i \in V$. The eigenvalues of a graph are the eigenvalues of its adjacency matrix. The *Seidel eigenvalues* of a graph are the eigenvalues of its Seidel

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matrix and are denoted by $\theta_1, \theta_2, \ldots, \theta_n$. If all the Seidel eigenvalues are integers, then the corresponding graph is called *Seidel integral* graph. The *Seidel energy* of G is defined as $\mathcal{E}_S(G) = \sum_{j=1}^{n} |\theta_j|$. Two graphs G_1 and G_2 with the same number of vertices are said to be Seidel equienergetic if $\mathcal{E}_S(G_1) = \mathcal{E}_S(G_2)$. Let n_S^-, n_S^0 and n_S^+ respectively, denote the number of negative, zero and positive Seidel eigenvalues of G. Let the graphs K_n and K_{n_1,n_2} denote the complete graph with n vertices and the complete bipartite graph with $n_1 + n_2$ vertices respectively. For other notation, terminology and the results related to the spectra of graphs, we follow [4].

Definition 2.1. [7] The *line graph* $\mathcal{L}(G)$ of a graph G is the graph with vertex set same as the edge set of G in which two vertices are adjacent if and only if the corresponding edges in G have a vertex in common. The *k-th iterated line graph* of G for $k = 0, 1, 2, \ldots$ is defined as $\mathcal{L}^k(G) \equiv \mathcal{L}(\mathcal{L}^{k-1}(G))$, where $\mathcal{L}^0(G) \equiv G$ and $\mathcal{L}^1(G) \equiv \mathcal{L}(G)$.

Definition 2.2. [9] Let the vertex set of a graph G be $V(G) = \{v_1, v_2, \ldots, v_n\}$. For $k \geq 2$, the *k-fold graph* $D_k[G]$ of a graph G is obtained by taking k copies of G in which a vertex v_i in one copy is adjacent to a vertex v_j in other copies if and only if v_i is adjacent v_j in G.

It is noted that the adjacency matrix of $D_k[G]$ is $A(D_k[G])$=$J_k \otimes A(G)$, where $\otimes$ denotes the Kronecker product. If $k = 2$, we get the *double graph* $D(G)$ [10], that is, $D_2[G] \equiv D(G)$.

Definition 2.3. Let the vertex set of a graph G be $V(G) = \{v_1, v_2, \ldots, v_n\}$. For $k \geq 2$, the *strong k-fold graph* $Sd_k[G]$ of a graph G is obtained by taking k copies of G in which a vertex v_i in one copy is adjacent to a vertex v_j in other copies if and only if v_i is adjacent v_j in G or $i = j$.

It is noted that the adjacency matrix of $Sd_k[G]$ is $A(Sd_k[G])$=$J_k \otimes (A(G) + I) - I \otimes I$. If $k = 2$, we get the *strong double graph* $Sd(G)$ [10, 12], that is, $Sd_2[G] \equiv Sd(G)$.

Lemma 2.4. [19] *Let the Seidel eigenvalues of a graph G with n vertices be θ_j, $1 \leq j \leq n$. Then for $k \geq 2$, the Seidel eigenvalues of $D_k[G]$ are $k\theta_j + (k-1)$, $1 \leq j \leq n$ and $-1\,(nk - n$ times).*

Lemma 2.5. [19] *Let the Seidel eigenvalues of a graph G with n vertices be θ_j, $1 \leq j \leq n$. Then for $k \geq 2$, the Seidel eigenvalues of $Sd_k[G]$ are $k\theta_j - (k-1)$, $1 \leq j \leq n$ and $1\,(nk - n$ times).*

Theorem 2.6. [3] *Let the eigenvalues of an r-regular graph G with n vertices be r, λ_i, $2 \leq i \leq n$. Then the Seidel eigenvalues of G are $n - 2r - 1$ and $-1 - 2\lambda_i$, $2 \leq i \leq n$.*

Theorem 2.7. [15] *Let G be a graph with n_0 number of vertices and m_0 number of edges such that $d_i + d_j \geq 6$ to each edge $e = v_iv_j$ in G. Then the iterated line graphs $\mathcal{L}^k(G)$ have all the negative eigenvalues equal to -2 with the multiplicity $m_{k-1} - n_{k-1}$ for $k \geq 2$, where n_k and m_k denote the number of vertices and the number of edges of $\mathcal{L}^k(G)$ respectively.*

Theorem 2.8. [16] *Let the graphs G_1 and G_2 be r-regular with the same number of vertices n and $r \geq 3$. Then $\mathcal{E}_S(\mathcal{L}^k(G_1)) = \mathcal{E}_S(\mathcal{L}^k(G_2))$ to each $k \geq 2$.*

3 Main Results

In the following, we give an explicit expression for Seidel energy of k-fold graph $D_k[G]$ in terms of Seidel energy of G for any graph G.

Let $n_\theta(\mathbf{I})$ denotes the number of Seidel eigenvalues of G which belongs to the interval $\mathbf{I}$ and let $\nu = 1 - \frac{1}{k}$, $k \geq 2$.

Theorem 3.1. *Let the Seidel eigenvalues of G be θ_j, $1 \leq j \leq n$. Then for $k \geq 2$,*

$$\mathcal{E}_S(D_k[G]) = k\left(2n\nu + \mathcal{E}_S(G) - 2\nu n_S^- + 2\sum_{\theta_j \in (-\nu, 0)} (\theta_j + \nu)\right).$$

Proof. Let $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ be the Seidel eigenvalues of G. By definition of Seidel energy of a graph, we have

$$\begin{aligned}
\mathcal{E}_S(D_k[G]) &= nk - n + \sum_{j=1}^{n} |k\theta_j + (k-1)| \quad \text{by Lemma 2.4} \\
&= kn\nu + k\sum_{j=1}^{n} |\nu + \theta_j| \\
&= k\Big(n\nu + \sum_{\theta_j \leq -\nu} (-\nu - \theta_j) + \sum_{\theta_j > -\nu} (\nu + \theta_j)\Big) \\
&= k\Big(n\nu - \nu n_\theta([\theta_n, -\nu]) + \sum_{\theta_j \leq -\nu} |\theta_j| + \nu n_\theta((-\nu, \theta_1]) + \sum_{\theta_j \in (-\nu,\, 0)} \theta_j + \sum_{\theta_j \geq 0} \theta_j\Big),
\end{aligned}$$

where $n_\theta([\theta_n, -\nu]) = 0$ if $\theta_n \geq -\nu$. The Seidel energy of a graph G can be expressed as

$$\mathcal{E}_S(G) = \sum_{j=1}^{n} |\theta_j| = \sum_{\theta_j \leq -\nu} |\theta_j| + \sum_{\theta_j \in (-\nu,\, 0)} |\theta_j| + \sum_{\theta_j \geq 0} \theta_j, \text{ with this we get}$$

$$\begin{aligned}
\mathcal{E}_S(D_k[G]) &= k\Big(n\nu - \nu n_\theta([\theta_n, -\nu]) + \nu n_\theta((-\nu, \theta_1]) + \sum_{\theta_j \in (-\nu,\, 0)} \theta_j + \mathcal{E}_S(G) \\
&\qquad - \sum_{\theta_j \in (-\nu,\, 0)} |\theta_j|\Big) \\
&= k\Big(n\nu - \nu n_\theta([\theta_n, -\nu]) + \nu n - \nu n_\theta([\theta_n, -\nu]) + 2\sum_{\theta_j \in (-\nu,\, 0)} \theta_j + \mathcal{E}_S(G)\Big) \\
&= k\Big(2n\nu + \mathcal{E}_S(G) - 2\big(\nu n_\theta([\theta_n, -\nu]) - \sum_{\theta_j \in (-\nu,\, 0)} \theta_j\big)\Big). \qquad (1)
\end{aligned}$$

The total number of Seidel eigenvalues n of a graph G can be expressed as

$$\begin{aligned}
n &= n_\theta([\theta_n, -\nu]) + n_\theta((-\nu, 0)) + n_S^0 + n_S^+ \quad \text{or} \\
n_\theta([\theta_n, -\nu]) &= n - n_S^+ - n_S^0 - n_\theta(-\nu, 0) = n_S^- - n_\theta((-\nu, 0)). \qquad (2)
\end{aligned}$$

Also, we have

$$\sum_{\theta_j \in (-\nu,\, 0)} (\theta_j + \nu) = \sum_{\theta_j \in (-\nu, 0)} \theta_j + \nu n_\theta((-\nu,\, 0)). \qquad (3)$$

Using (2) and (3) in (1), we get

$$\mathcal{E}_S(D_k[G]) = k\big(2n\nu + \mathcal{E}_S(G) - 2\nu n_S^- + 2\sum_{\theta_j \in (-\nu,\, 0)} (\theta_j + \nu)\big)$$

which completes the proof. □

It is easy to observe that to each negative Seidel eigenvalue $\theta_j \in (-\nu, 0)$ we have $0 < \theta_j + \nu < \nu$, which gives $\nu n_S^- > \sum_{\theta_j \in (-\nu,\, 0)} (\theta_j + \nu) > 0$ for any graph G. Using this fact we get the following.

Corollary 3.2. *Let G be a graph with n vertices. Then for $k \geq 2$,*

$$2n(k-1) + k\mathcal{E}_S(G) - 2n_S^-(k-1) \leq \mathcal{E}_S(D_k[G]) < 2n(k-1) + k\mathcal{E}_S(G).$$

□

It is noted that $(-\nu, 0) \subseteq (-1, 0)$ for $k \geq 2$. There are many graphs with no Seidel eigenvalues in the interval $(-1, 0)$, for instance, all Seidel integral graphs. If a graph G has no Seidel eigenvalue in the interval $(-\nu, 0)$ then we have the following.

Corollary 3.3. *Let G be a graph with n vertices. Then for $k \geq 2$, G has no Seidel eigenvalue in the interval $(-\nu, 0)$ if and only if*

$$\mathcal{E}_S(D_k[G]) = 2(k-1)(n - n_S^-) + k\mathcal{E}_S(G).$$

Proof. Proof follows directly from the fact that $\sum\limits_{\theta \in (-\nu, 0)} (\theta + \nu) = 0$ if and only if G has no Seidel eigenvalue θ in the interval $(-\nu, 0)$ in the Theorem 3.1. □

It is easy to construct Seidel equienergetic graphs by using Theorem 3.1 with the help of Seidel equienergetic graphs with no Seidel eigenvalues in the interval $(-\nu, 0)$ and having the same number of negative Seidel eigenvalues.
Let the Seidel eigenvalues of two graphs G_1 and G_2 be $\theta_1^{'}, \theta_2^{'}, \ldots, \theta_n^{'}$ and $\theta_1^{''}, \theta_2^{''}, \ldots, \theta_n^{''}$ and let the number of negative Seidel eigenvalues of G_1 and G_2 be n_{S1}^- and n_{S2}^- respectively.

Corollary 3.4. *Let G_1 and G_2 be Seidel equienergetic graphs with n vertices. Then for $k \geq 2$, the graphs $D_k[G_1]$ and $D_k[G_2]$ are Seidel equienergetic if and only if* $\nu n_{S1}^- - \sum\limits_{\theta_j^{'} \in (-\nu, 0)} (\theta_j^{'} + \nu) = \nu n_{S2}^- - \sum\limits_{\theta_j^{''} \in (-\nu, 0)} (\theta_j^{''} + \nu)$.
In particular, if G_1 and G_2 have no Seidel eigenvalues in the interval $(-\nu, 0)$ then for $k \geq 2$, the graphs $D_k[G_1]$ and $D_k[G_2]$ are Seidel equienergetic if and only if $n_{S1}^- = n_{S2}^-$. □

Example 3.5. The graphs $\mathcal{L}^p(K_{n,n} \Box K_{n-1})$ and $\mathcal{L}^p(K_{n-1,n-1} \Box K_n)$ are integral Seidel equienergetic graphs with the same number of negative Seidel eigenvalues for all $n \geq 5, p \geq 0$ [13], where $\Box$ denotes the Cartesian product. Therefore by Corollary 3.4, the graphs $D_k[\mathcal{L}^p(K_{n,n} \Box K_{n-1})]$ and $D_k[\mathcal{L}^p(K_{n-1,n-1} \Box K_n)]$ are Seidel equienergetic for all $k \geq 2, n \geq 5$ and $p \geq 0$.

There are many non-isomorphic regular graphs with same number of vertices and same degree, see [8, 13, 14, 17, 18]. Ramane et al. in [16] shown a way to construct a large pairs of Seidel equienergetic iterated line graphs by using such regular graphs. In the following, we present another large class of Seidel equienergetic graphs.

Theorem 3.6. *Let the graphs G_1 and G_2 be two r-regular Seidel equienergetic graphs with same number of vertices n and $r \geq 3$. Then the graphs $D_k[\mathcal{L}^p(G_1)]$ and $D_k[\mathcal{L}^p(G_2)]$ are Seidel equienergetic for all $k \geq 2$ and $p \geq 2$.*

Proof. If $r \geq 3$ for an r-regular graph G, then the iterated line graphs $\mathcal{L}^p(G)$ are also regular. By Theorem 2.7, the graphs $\mathcal{L}^p(G)$, $p \geq 2$ have all negative eigenvalues equal to -2. Now using the Theorem 2.6, it is evident that all the negative Seidel eigenvalues of $\mathcal{L}^p(G)$, $p \geq 2$ are less than or equal to -1. Therefore, if the graphs G_1 and G_2 are two r-regular graphs with same number of vertices n and $r \geq 3$ then the graphs $\mathcal{L}^p(G_1)$ and $\mathcal{L}^p(G_2)$ have no Seidel eigenvalues in the interval $(-1, 0)$ to each $p \geq 2$. Also the graphs $\mathcal{L}^p(G_1)$ and $\mathcal{L}^p(G_2)$ are Seidel equienergetic by Theorem 2.8. Hence by Corollary 3.4 the graphs $D_k[\mathcal{L}^p(G_1)]$ and $D_k[\mathcal{L}^p(G_2)]$ are Seidel equienergetic for all $k \geq 2$ and $p \geq 2$. □

It is interesting to see the Seidel eigenvalues of $D_k[G]$ of a graph G in the interval $(-1, 0)$.

Proposition 3.7. *If a graph G has no Seidel eigenvalues in the interval $(-1, 0)$, then for $k \geq 2$, $D_k[G]$ also have no Seidel eigenvalues in the interval $(-1, 0)$.*

Proof. Proof follows directly from the Seidel eigenvalues of $D_k[G]$ in the Lemma 2.4 if G has no Seidel eigenvalues in the interval $(-1, 0)$. □

In the following, we give an explicit expression for Seidel energy of strong k-fold graph $Sd_k[G]$, $k \geq 2$ in terms of Seidel energy of G for any graph G.

Theorem 3.8. *Let the Seidel eigenvalues of G be θ_j, $1 \leq j \leq n$. If $\theta_j \notin (-\nu, \nu)$ then for $k \geq 2$,*

$$\mathcal{E}_S(Sd_k[G]) = 2(k-1)(n - n_S^+) + k\mathcal{E}_S(G).$$

Proof. Let $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ be the Seidel eigenvalues of G. If $\theta_j \notin (-\nu, \nu)$, then we have

$$|k\theta_j - (k-1)| = \begin{cases} k|\theta_j| - (k-1) & \text{if } \theta_j \geq \nu \\ k|\theta_j| + (k-1) & \text{if } \theta_j \leq -\nu \ . \end{cases}$$

By definition of Seidel energy of a graph, we have

$$\begin{aligned}
\mathcal{E}_S(Sd_k[G]) &= nk - n + \sum_{j=1}^{n} |k\theta_j - (k-1)| \quad \text{by Lemma 2.5} \\
&= n(k-1) + \sum_{\theta_j \leq -\nu} (k|\theta_j| + (k-1)) + \sum_{\theta_j \geq \nu} (k|\theta_j| - (k-1)) \\
&= n(k-1) + k \sum_{\theta_j \leq -\nu} |\theta_j| + (k-1) n_\theta([\theta_n, -\nu]) + k \sum_{\theta_j \geq \nu} |\theta_j| \\
&\quad - (k-1) n_\theta([\nu, \theta_1]) \\
&= n(k-1) + k\mathcal{E}_S(G) + (k-1)(n_\theta([\theta_n, -\nu]) - n_\theta([\nu, \theta_1])).
\end{aligned}$$

If If $\theta_j \notin (-\nu, \nu)$, then total number of Seidel eigenvalues n of a graph G can be expressed as $n = n_\theta([\theta_n, -\nu]) + n_\theta([\nu, \theta_1])$, with this fact we have

$$\begin{aligned}
\mathcal{E}_S(Sd_k[G]) &= n(k-1) + k\mathcal{E}_S(G) + (k-1)(n - n_\theta([\nu, \theta_1]) - n_\theta([\nu, \theta_1])) \\
&= 2n(k-1) + k\mathcal{E}_S(G) - 2(k-1)(n_\theta([\nu, \theta_1])). \\
&= 2n(k-1) + k\mathcal{E}_S(G) - 2(k-1)n_S^+, \quad \text{since } \nu > 0 \text{ and } \theta_j \notin (-\nu, \nu) \\
&= 2(k-1)(n - n_S^+) + k\mathcal{E}_S(G).
\end{aligned}$$

which completes the proof. □

In the following, another class of Seidel equienergetic graphs are characterized. Let the number of positive Seidel eigenvalues of the graphs G_1 and G_2 be n_{S1}^+ and n_{S2}^+ respectively.

Corollary 3.9. *Let G_1 and G_2 be Seidel equienergetic graphs with no Seidel eigenvalues in the interval $(-\nu, \nu)$ and both with n vertices. Then for $k \geq 2$, the graphs $Sd_k[G_1]$ and $Sd_k[G_2]$ are Seidel equienergetic if and only if $n_{S1}^+ = n_{S2}^+$.* □

Example 3.10. The graphs $K_{n,n} \boxtimes K_{n-1}$ and $K_{n-1,n-1} \boxtimes K_n$ are integral Seidel equienergetic graphs with the same number of positive Seidel eigenvalues for all $n \geq 3$ [13], where $\boxtimes$ denotes the strong product. Therefore by Corollary 3.9, the graphs $Sd_k[K_{n,n} \boxtimes K_{n-1}]$ and $Sd_k[K_{n-1,n-1} \boxtimes K_n]$ are Seidel equienergetic for all $n \geq 3$ and $k \geq 2$.

The following is Theorem 2.4 of [19] which is the consequence of Corollary 3.3 and Theorem 3.8.

Theorem 3.11. *Let the Seidel eigenvalues of G be θ_j, $1 \leq j \leq n$ and $\theta_j \notin (-\nu, \nu)$. Then for $k \geq 2$ the graphs $D_k[G]$ and $Sd_k[G]$ are Seidel equienergetic if and only if $n_S^- = n_S^+$.* □

In the following, we present the Seidel energy of $Sd_k[D_k[G]]$, $k \geq 2$ for any graph G.

Theorem 3.12. *Let the Seidel eigenvalues of G be θ_j, $1 \leq j \leq n$. Then for $k \geq 2$,*

$$\mathcal{E}_S(Sd_k[D_k[G]]) = 2n(k-1)(2k-1) + k^2\big(\mathcal{E}_S(G) - 2\nu^2 n_S^- + 2 \sum_{\theta_j \in (-\nu^2, 0)} (\theta_j + \nu^2)\big).$$

Proof. If $\theta_1, \theta_2, \ldots, \theta_n$ are the Seidel eigenvalues of G, then by Lemma 2.4 and Lemma 2.5, the Seidel eigenvalues of $Sd_k[D_k[G]]$ are $k^2\theta_j + (k-1)^2$, $1 \leq j \leq n$, $1-2k$ ($nk-n$ times) and 1 ($nk^2 - nk$ times) [19]. By definition of Seidel energy of a graph, we have

$$\begin{aligned}\mathcal{E}_S(Sd_k[D_k[G]]) &= nk^2 - nk + (2k-1)(nk-n) + \sum_{j=1}^{n} |k^2\theta_j + (k-1)^2| \\ &= 3nk^2 - 4kn + n + \sum_{j=1}^{n} |k^2\theta_j + (k-1)^2|\end{aligned}$$

Now proceeding similar to that of proof of Theorem 3.1, we get

$$\mathcal{E}_S(Sd_k[D_k[G]]) = 2n(k-1)(2k-1) + k^2\big(\mathcal{E}_S(G) - 2\nu^2 n_S^- + 2\sum_{\theta_j \in (-\nu^2,\, 0)} (\theta_j + \nu^2)\big),$$

which completes the proof. □

Again, it can be seen that to each negative Seidel eigenvalue $\theta_j \in (-\nu^2, 0)$ we have $0 < \theta_j + \nu^2 < \nu^2$, which gives $\nu^2 n_S^- > \sum_{\theta_j \in (-\nu^2,\, 0)} (\theta_j + \nu^2) > 0$ for any graph G. Using this fact we get the following.

Corollary 3.13. *Let G be a graph with n vertices. Then for $k \geq 2$,*

$$2n(k-1)(2k-1) + k^2\mathcal{E}_S(G) - 2(k-1)^2 n_S^- \leq \mathcal{E}_S(Sd_k[D_k[G]]) < 2n(k-1)(2k-1) + k^2\mathcal{E}_S(G).$$

□

Again, it is noted that $(-\nu^2, 0) \subseteq (-1, 0)$ for $k \geq 2$. If a graph G has no Seidel eigenvalue in the interval $(-\nu^2, 0)$ then we have the following.

Corollary 3.14. *Let G be a graph with n vertices. Then for $k \geq 2$, G has no Seidel eigenvalue in the interval $(-\nu^2, 0)$ if and only if*

$$\mathcal{E}_S(Sd_k[D_k[G]]) = 2n(k-1)(2k-1) + k^2\mathcal{E}_S(G) - 2(k-1)^2 n_S^-.$$

Proof. Proof follows directly from the fact that $\sum_{\theta \in (-\nu^2, 0)} (\theta + \nu^2) = 0$ if and only if G has no Seidel eigenvalue θ in the interval $(-\nu^2, 0)$ in the Theorem 3.12. □

The following provides a way to construct Seidel equienergetic graphs. Let the Seidel eigenvalues of two graphs G_1 and G_2 be $\theta_1', \theta_2', \ldots, \theta_n'$ and $\theta_1'', \theta_2'', \ldots, \theta_n''$ and let the number of negative Seidel eigenvalues of G_1 and G_2 be n_{S1}^- and n_{S2}^- respectively.

Corollary 3.15. *Let G_1 and G_2 be Seidel equienergetic graphs with n vertices. Then for $k \geq 2$, the graphs $Sd_k[D_k[G_1]]$ and $Sd_k[D_k[G_2]]$ are Seidel equienergetic if and only if $\nu^2 n_{S1}^- - \sum_{\theta_j' \in (-\nu^2, 0)} (\theta_j' + \nu^2) = \nu^2 n_{S2}^- - \sum_{\theta_j'' \in (-\nu^2, 0)} (\theta_j'' + \nu^2)$. In particular, if G_1 and G_2 have no Seidel eigenvalues in the interval $(-\nu^2, 0)$ then for $k \geq 2$, the graphs $Sd_k[D_k[G_1]]$ and $Sd_k[D_k[G_2]]$ are Seidel equienergetic if and only if $n_{S1}^- = n_{S2}^-$.* □

Example 3.16. Consider the graphs $\mathcal{L}^p(K_{n,n} \Box K_{n-1})$ and $\mathcal{L}^p(K_{n-1,\, n-1} \Box K_n)$ in the example 3.5. By using Corollary 3.15, the graphs $Sd_k[D_k[\mathcal{L}^p(K_{n,n} \Box K_{n-1})]]$ and $Sd_k[D_k[\mathcal{L}^p(K_{n-1,\, n-1} \Box K_n)]]$ are Seidel equienergetic for all $k \geq 2, n \geq 5$ and $p \geq 0$.

In the following, we present another large class of Seidel equienergetic graphs.

Theorem 3.17. *Let the graphs G_1 and G_2 be two r-regular Seidel equienergetic graphs with same number of vertices n and $r \geq 3$. Then the graphs $Sd_k[D_k[\mathcal{L}^p(G_1)]]$ and $Sd_k[D_k[\mathcal{L}^p(G_2)]]$ are Seidel equienergetic for all $k \geq 2$ and $p \geq 2$.*

Proof. Proof follows similar to that of proof of Theorem 3.6 with the help of Corollary 3.15. □

In the following, we present the Seidel energy of $D_k[Sd_k[G]]$, $k \geq 2$ for any graph G.

Theorem 3.18. *Let the Seidel eigenvalues of G be θ_j, $1 \leq j \leq n$. If $\theta_j \notin (-\nu^2, \nu^2)$ then for $k \geq 2$,*

$$\mathcal{E}_S(D_k[Sd_k[G]]) = 2n(k-1)(2k-1) + k^2\mathcal{E}_S(G) - 2(k-1)^2 n_S^+.$$

Proof. If $\theta_1, \theta_2, \ldots, \theta_n$ are the Seidel eigenvalues of G, then by Lemma 2.4 and Lemma 2.5, the Seidel eigenvalues of $D_k[Sd_k[G]]$ are $k^2\theta_j - (k-1)^2$, $1 \leq j \leq n$, $2k-1$ ($nk-n$ times) and -1 ($nk^2 - nk$ times) [19]. By definition of Seidel energy of a graph, we have

$$\begin{aligned}\mathcal{E}_S(D_k[Sd_k[G]]) &= nk^2 - nk + (2k-1)(nk-n) + \sum_{j=1}^{n} |k^2\theta_j - (k-1)^2| \\ &= 3nk^2 - 4kn + n + \sum_{j=1}^{n} |k^2\theta_j - (k-1)^2|\end{aligned}$$

Now proceeding similar to that of proof of Theorem 3.8, we get

$$\mathcal{E}_S(D_k[Sd_k[G]]) = 2n(k-1)(2k-1) + k^2\mathcal{E}_S(G) - 2(k-1)^2 n_S^+,$$

which completes the proof. □

In the following, we present another class of Seidel equienergetic graphs. Let the number of positive Seidel eigenvalues of the graphs G_1 and G_2 be n_{S1}^+ and n_{S2}^+ respectively.

Corollary 3.19. *Let G_1 and G_2 be Seidel equienergetic graphs with no Seidel eigenvalues in the interval $(-\nu^2, \nu^2)$ and both with n vertices. Then for $k \geq 2$, the graphs $D_k[Sd_k[G_1]]$ and $D_k[Sd_k[G_2]]$ are Seidel equienergetic if and only if $n_{S1}^+ = n_{S2}^+$.* □

Example 3.20. Consider the graphs $K_{n,n} \boxtimes K_{n-1}$ and $K_{n-1,n-1} \boxtimes K_n$ in the example 3.10. Now by using the Corollary 3.19, the graphs $D_k[Sd_k[K_{n,n} \boxtimes K_{n-1}]]$ and $D_k[Sd_k[K_{n-1,n-1} \boxtimes K_n]]$ are Seidel equienergetic for all $n \geq 3$ and $k \geq 2$.

The following is Theorem 2.5 of [19] which is the consequence of Corollary 3.14 and Theorem 3.18.

Theorem 3.21. *Let the Seidel eigenvalues of G be θ_j, $1 \leq j \leq n$ and $\theta_j \notin (-\nu^2, \nu^2)$. Then for $k \geq 2$ the graphs $Sd_k[D_k[G]]$ and $D_k[Sd_k[G]]$ are Seidel equienergetic if and only if $n_S^- = n_S^+$.* □

4 Conclusion

Vaidya and Popat in [19] constructed Seidel equienergetic graphs by using the graphs $D_k[G]$ and $Sd_k[G]$ for any graph G, where $k \geq 2$. In this paper, we have given the explicit expressions for the Seidel energy of the graphs $D_k[G]$ and $Sd_k[G]$ and provided a general way to construct certain class of Seidel equienergetic graphs.

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Email: `hsramane@yahoo.com`
Email: `bparvathalu@gmail.com`
Email: `ashokagonal@gmail.com`