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Motzkin prefix numbers

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Abstract

Consider an $m \times n$ table T and latices paths $\nu_1, \ldots, \nu_k$ in T such that each step $\nu_{i+1} - \nu_i = (1,1)$, $(1,0)$ or $(1,-1)$. The number of paths from the $(1,i)$-cell (resp. first column) to the (s,t)-cell is denoted by $\mathcal{D}^i(s,t)$ (resp. $\mathcal{D}(s,t)$). Also, the number of all paths form the first column to the last column is denoted by $\mathcal{I}_m(n)$. We give explicit formulas for the numbers $\mathcal{D}^1(s,t)$ and $\mathcal{D}(s,t)$. As a result, we prove a conjecture of *Alexander R. Povolotsky* involving $\mathcal{I}_n(n)$. Finally, we present some relationships between the number of lattice paths and Fibonacci and Pell-Lucas numbers, and pose several open problems.

Keywords: Direct animals, Lattice paths, Dyck paths, Perfect lattice paths, Ballot numbers, Motzkin numbers.
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1 Introduction

A lattice path in $\mathbb{Z}^2$ is the drawing in $\mathbb{Z}^2$ of a sum of vectors from a fixed finite subset S of $\mathbb{Z}^2$, starting from a given point, say $(0,0)$ of $\mathbb{Z}^2$. A typical problem in lattice paths is the enumeration of all $\mathbf{S}$-lattice paths (lattice paths with respect to the set $\mathbf{S}$) with a given initial and terminal point satisfying possibly some further constraints. A nontrivial simple case is the problem of finding the number of lattice paths starting from the origin $(0,0)$ and ending at a point (m,n) using only right step $(1,0)$ and up step $(0,1)$ (i.e., $\mathbf{S} = \{(1,0),(0,1)\}$). The number of such paths are known to be the the binomial coefficient $\binom{m+n}{n}$. Yet another example, known as the ballot problem, is to find the number of lattice paths from $(1,0)$ to (m,n) with $m > n$, using the same steps as above, that never touch the line $y = x$. The number of such paths, known as ballot number, equals $\frac{m-n}{m+n}\binom{m+n}{n}$. In the special case where $m = n+1$, the ballot number is indeed the Catalan number C_n.

Let $T_{m,n}$ denote the $m \times n$ table in the plane and (x,y) be the cell in the columns x and row y (and refer to it as the (x,y)-cell). The set of lattice paths from the (i,j)-cell to the (s,t)-cell, with steps belonging to a finite set $\mathbf{S}$, is denoted by $L((i,j) \to (s,t); \mathbf{S})$, and the number of those paths is denoted by $L((i,j) \to (s,t); \mathbf{S})$, where $1 \leqslant i,s \leqslant m$ and $1 \leqslant j,t \leqslant n$. We put $|L((i,j) \to (s,t); \mathbf{S})| = l((i,j) \to (s,t); \mathbf{S})$ which means the number of all lattice paths from the (i,j)-cell to the (s,t)-cell.

Throughout this paper, for the table $T_{m,n}$, we set $\mathbf{S} = \{(1,1),(1,0),(1,-1)\}$, and the corresponding lattice paths starting from the first column and ending at the last column are called *perfect lattice paths*. The number of all perfect lattice paths is denoted by $\mathcal{I}_m(n)$, that is,

$$\mathcal{I}_m(n) = \sum_{i,j=1}^{m} l((i,j) \to (s,t); \mathbf{S}).$$

[1]speaker

The values of $\mathcal{I}_m(n)$ is OEIS sequence A081113 and A296449. Figure 1 shows the number of all lattice paths for $m = 2$ and $n = 3$. Clearly, $l((1, i) \to (n, j)) = l((1, i') \to (n, j'))$ when $i + i' = m + 1$ and $j + j' = m + 1$.

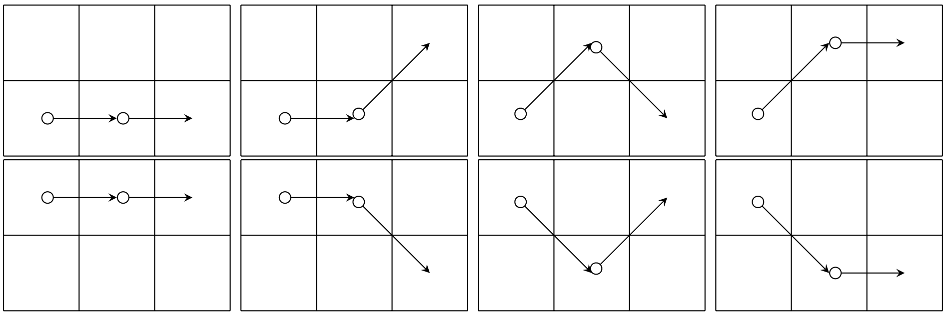

Figure 1: All lattice paths in $T_{2,3}$.

In this paper, by using ballot numbers, we give explicit formulas for the numbers $\mathcal{D}^1(s, t)$ and $\mathcal{D}(s, t)$ where are defined in the section 2. We closed this paper by several interesting conjectures and problems.

2 $\mathcal{I}_n(n)$ vs Alexander R. Povolotsky's conjecture

Let $\mathbf{S} := \{(1, 1), (1, 0), (1, -1)\}$. For positive integers $1 \leqslant i, t \leqslant m$ and $1 \leqslant s \leqslant n$, the number of all lattice paths from the $(1, i)$-cell to the (s, t)-cell in the table $T = T_{m,n}$ is denoted by $\mathcal{D}^i(s, t)$, that is, $\mathcal{D}^i(s, t) = l((1, i) \to (s, t); \mathbf{S})$. We put

$$\mathcal{D}_{m,n}(s, t) = \sum_{i=1}^{m} \mathcal{D}^i(s, t).$$

For the case $n = m$, we show these numbers by $\mathcal{D}_n(s, t) := \mathcal{D}_{n,n}(s, t)$, that called the n-th Motzkin prefix number is the number of three-step paths consisting of n steps, starting at the origin, and not running below the x-axis (with any end point). Clearly, $\mathcal{D}(s, t)$ is the number of all lattice paths from first column to the (s, t)-cell of T. It is easy to see for $n \geqslant 2$

$$\mathcal{D}_n(n, n) = \mathcal{D}_n(n - 1, n) + \mathcal{D}_n(n - 1, n - 1),$$

where $\mathcal{D}_1(1, 1) = 1, \mathcal{D}_2(2, 2) = 2, \mathcal{D}_3(3, 3) = 5, \mathcal{D}_4(4, 4) = 13, \ldots$. The values of $\mathcal{D}_n(n, n)$ is OEIS sequence A005773, where T is a square table. By the way, By the way, notice how the diagram for $\mathcal{D}_4(4, 4) = 13$ is

1	2	5	13
1	3	8	21
1	3	8	21
1	2	5	13

where each entry is the sum of two or three entries in the preceding column.

By symmetry of the table T, we have $\mathcal{D}(s, t) = \mathcal{D}(s, t')$ when $t + t' = m + 1$. Table 1 illustrates the values of $\mathcal{D}(6, t)$, for all $1 \leqslant t \leqslant 6$, where the number in (s, t)-cell of T determines the number $\mathcal{D}(s, t)$.

It is worth mentioning that the numbers $\mathcal{D}_n(n, n)$ coincide with the number of directed animals of size n starting from a single point (see [24]). The numbers $\mathcal{D}_n(n, n)$ appear is various other results, see for example [9, 11, 12, 15, 19]. Note also that Krattenthaler and Yaqubi [32] compute determinants of some Hankel matrices involving $\mathcal{D}_n(x, y)$, which is of independent interest.

Theorem 2.1. *For any positive integer n we have*

$$\mathcal{I}_n(n) = 3\mathcal{I}_{n-1}(n - 1) + 3^{n-1} - 2\mathcal{D}_{n-1}(n - 1, n - 1).$$

					$\mathcal{D}(6,t)$
1	2	5	13	35	96
1	3	8	22	61	170
1	3	9	26	74	209
1	3	9	26	74	209
1	3	8	22	61	170
1	2	5	13	35	96

Table 1: Values of $\mathcal{D}(6,t)$

Proof. Let $T := T_{n,n}$ and $T' := T_{n-1,n-1}$ with T' in the left-bottom side of T. Clearly, the number of lattice paths of T which never meet the n^{th} row of T is

$$\mathcal{I}_{n-1}(n) = 3\mathcal{I}_{n-1}(n-1) - 2\mathcal{D}_{n-1}(n-1,n-1).$$

To obtain the number of all lattice paths we must count those who meet the n^{th}-row of T, that is equal to 3^{n-1}. Thus $\mathcal{I}_n(n) - \mathcal{I}_{n-1}(n) = 3^{n-1}$, from which the result follows. □

Michael Somos in OEIS sequence A005773 gives the following recurrence relation for $\mathcal{D}_n(n,n)$.

Theorem 2.2. *Inside the square $n \times n$ table we have*

$$n\mathcal{D}_n(n,n) = 2n\mathcal{D}_n(n-1,n-1) + 3(n-2)\mathcal{D}_n(n-2,n-2).$$

Utilizing Theorems 2.1 and 2.2 for $\mathcal{D}_n(n,n)$, we can prove a conjecture of Alexander R. Povolotsky posed in OEIS sequence A081113 as follows. This identity has appeared first in [6]

Theorem 2.3 (Alexander R. Povolotsky Conjecture)**.** *The following identity holds for the numbers $\mathcal{I}_n(n)$.*

$$\begin{aligned}(n+3)\mathcal{I}_{n+4}(n+4) &= 27n\mathcal{I}_n(n) + 27\mathcal{I}_{n+1}(n+1)\\ &- 9(2n+5)\mathcal{I}_{n+2}(n+2) + (8n+21)\mathcal{I}_{n+3}(n+3).\end{aligned}$$

Proof. Put

$$\begin{aligned}A &= (n+3)\mathcal{I}_{n+4}(n+4),\\ B &= (8n+21)\mathcal{I}_{n+3}(n+3),\\ C &= 9(2n+5)\mathcal{I}_{n+2}(n+2),\\ D &= 27\mathcal{I}_{n+1}(n+1),\\ E &= 27n\mathcal{I}_n(n).\end{aligned}$$

Using Theorem 2.1, we can write

$$\begin{aligned}A =&(3n+9)\mathcal{I}_{n+3}(n+3) + (n+3)3^{n+3} - (2n+6)\mathcal{D}_n(n+3,n+3)\\ =&(8n+21)\mathcal{I}_{n+3}(n+3) - (5n+12)\mathcal{I}_{n+3}(n+3) + (n+3)3^{n+3}\\ &- (2n+6)\mathcal{D}_n(n+3,n+3)\\ =&B + (n+3)3^{n+3} - (5n+12)\mathcal{I}_{n+3}(n+3)\\ &- (2n+6)\mathcal{D}_n(n+3,n+3).\end{aligned} \tag{1}$$

Utilizing Theorem 2.1 once more for $\mathcal{I}_{n+3}(n+3)$ and $\mathcal{I}_{n+2}(n+2)$ yields

$$\begin{aligned}A =&B + (n+3)3^{n+3} - (5n+12)3^{n+2}\\ &- (18n+45)\mathcal{I}_{n+2}(n+2) - (2n+6)\mathcal{D}_n(n+3,n+3)\\ &+ (10n+24)\mathcal{D}_n(n+2,n+2) + (3n+9)\mathcal{I}_{n+2}(n+2) + (n+3)3^{n+3}\\ =&B - C - (5n+12)3^{n+2} - (2n+6)\mathcal{D}_n(n+3,n+3)\\ &+ (10n+24)\mathcal{D}_n(n+2,n+2) + 9n\mathcal{I}_{n+1}(n+1)\\ &+ 27\mathcal{I}_{n+1}(n+1) + (3n+9)3^{n+1} - (6n+18)\mathcal{D}_n(n+1,n+1).\end{aligned}$$

It can be easily shown that

$$\begin{aligned}A =& B - C + D \\ &+ (n+3)3^{n+3} - (2n+6)\mathcal{D}_n(n+3,n+3) - (5n+12)3^{n+2} \\ &+ (10n+24)\mathcal{D}_n(n+2,n+2) + 9n\mathcal{I}_{n+1}(n+1) \\ &+ (3n+9)3^{n+1} - (6n+18)\mathcal{D}_n(n+1,n+1).\end{aligned} \tag{2}$$

Replacing $9n\mathcal{I}_{n+1}(n+1)$ by $27n\mathcal{I}_n(n) + n3^{n+2} - 18n\mathcal{I}_n(n)$ in 2 gives

$$\begin{aligned}A = B - C + D + E \\ - (2n+6)\mathcal{D}_n(n+3,n+3) + (10n+24)\mathcal{D}_n(n+2,n+2) \\ - 18n\mathcal{D}_n(n,n) - (6n+18)\mathcal{D}_n(n+1,n+1).\end{aligned}$$

Since the coefficient of $\mathcal{D}_n(n+3,n+3)$ is $2(n+3)$, it follow from Theorem 2.2 that

$$\begin{aligned}A =& B - C + D + E - (4n+12)\mathcal{D}_n(n+2,n+2) - 18n\mathcal{D}_n(n,n) \\ &+ (10n+24)\mathcal{D}_n(n+2,n+2) - (6n+6)\mathcal{D}_n(n+1,n+1) \\ &- (6n+18)\mathcal{D}_n(n+1,n+1) \\ =& B - C + D + E - (4n+12)\mathcal{D}_n(n+2,n+2) \\ &- (6n+6)\mathcal{D}_n(n+1,n+1) + 18n\mathcal{D}_n(n,n) - 18n\mathcal{D}_n(n,n) \\ &- (12n+24)\mathcal{D}_n(n+1,n+1) + (6n+18)\mathcal{D}_n(n+1,n+1) \\ =& B - C + D + E,\end{aligned}$$

as required. □

3 Tables with few rows

In this section, we shall compute $\mathcal{I}_m(n)$ for $m = 1, 2, 3, 4$ and arbitrary positive integers n. Also, we obtain some properties of $\mathcal{I}_m(n)$ for $m = 5$. Some values of the $\mathcal{I}_3(n)$ and $\mathcal{I}_4(n)$ are already given in A001333 and A055819, respectively.

Lemma 3.1. *$\mathcal{I}_1(n) = 1$ and $\mathcal{I}_2(n) = 2^n$ for all $n \geqslant 1$.*

Let x and y be arbitrary real numbers. By the binomial theorem, we have the following identity,

$$x^n + y^n = (x+y)^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (xy)^k (x+y)^{n-2k},$$

where $n \geqslant 1$. This identity also can rewritten as

$$x^n + y^n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \left[\binom{n-k}{k} + \binom{n-k-1}{k-1} \right] (xy)^k (x+y)^{n-2k}, \tag{3}$$

where $\binom{r}{-1} = 0$. Pell-Lucas sequence [29] is defined as $\mathcal{Q}_1 = 1$, $\mathcal{Q}_2 = 3$, and $\mathcal{Q}_n = 2\mathcal{Q}_{n-1} + \mathcal{Q}_{n-2}$ for all $n \geqslant 3$. It can also be defined by the so called *Binet formula* as $\mathcal{Q}_n = (\alpha^n + \beta^n)/2$, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$ are solutions of the quadratic equation $x^2 = 2x + 1$.

Lemma 3.2. *For all $n \geqslant 1$ we have $\mathcal{I}_3(n) = \mathcal{Q}_{n+1}$.*

Proof. The number of lattice paths to cells in columns $n-2$, $n-1$, and n of $T_{3,n}$ looks like

$n-2$	$n-1$	n
x	$x+y$	$3x+2y$
y	$2x+y$	$4x+3y$
x	$x+y$	$3x+2y$

which imply that $\mathcal{I}_3(n-2) = 2x+y$, $\mathcal{I}_3(n-1) = 4x+3y$, and $\mathcal{I}_3(n) = 10x+7y$. Thus the following linear recurrence exists for $\mathcal{I}_3$.

$$\mathcal{I}_3(n) = 2\mathcal{I}_3(n-1) + \mathcal{I}_3(n-2). \tag{4}$$

Since $\mathcal{I}_3(1) = \mathcal{Q}_2 = 3$ and $\mathcal{I}_3(2) = \mathcal{Q}_3 = 7$, it follows that $\mathcal{I}_3(n) = \mathcal{Q}_{n+1}$ for all $n \geqslant 1$, as required. □

Corollary 3.3. *Let n be a positive integer. Then*

$$\mathcal{I}_3(n) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \left[\binom{n-k+1}{k} + \binom{n-k}{k-1} \right] 2^{n-2k}.$$

Proof. It is sufficient to put $x = \alpha$ and $y = \beta$ in (3). □

The Fibonacci sequence A000045 starts with the integers 0 and 1, and every other term is the sum of the two preceding ones, that is, $\mathcal{F}_0 = 0$, $\mathcal{F}_1 = 1$, and $\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2}$ for all $n \geqslant 2$. This recursion gives the Binet's formula $\mathcal{F}_n = \frac{\varphi^n - \psi^n}{\varphi - \psi}$, where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$.

Lemma 3.4. *For all $n \geqslant 1$ we have $\mathcal{I}_4(n) = 2\mathcal{F}_{2n+1}$.*

Proof. The number of lattice paths to cells in columns $n-2$, $n-1$, and n of $T_{4,n}$ looks like

$n-2$	$n-1$	n
x	$x+y$	$2x+3y$
y	$x+2y$	$3x+5y$
y	$x+2y$	$3x+5y$
x	$x+y$	$2x+3y$

which imply that $\mathcal{I}_4(n-2) = 2x+2y$, $\mathcal{I}_4(n-1) = 4x+6y$, and $\mathcal{I}_4(n) = 10x+16y$. Hence we get the following linear recurrence for $\mathcal{I}_4$.

$$\mathcal{I}_4(n) = 3\mathcal{I}_4(n-1) - \mathcal{I}_4(n-2). \tag{5}$$

On the other hand,

$$\begin{aligned} \mathcal{F}_{2n+1} &= \mathcal{F}_{2n} + \mathcal{F}_{2n-1} \\ &= 2\mathcal{F}_{2n-1} + \mathcal{F}_{2n-2} \\ &= 3\mathcal{F}_{2n-1} - \mathcal{F}_{2n-3} \\ &= 3\mathcal{F}_{2(n-1)+1} - \mathcal{F}_{2(n-2)+1}. \end{aligned}$$

Now since $\mathcal{I}_4(1) = 2\mathcal{F}_3$ and $\mathcal{I}_4(2) = 2\mathcal{F}_5$, it follows that $\mathcal{I}_4(n) = 2\mathcal{F}_{2n+1}$ for all $n \geqslant 1$. The proof is complete. □

Corollary 3.5. *For all $n \geqslant 1$ we have*

$$\mathcal{I}_4(n) = \sum_{k=0}^{n} (-1)^k \left[\frac{2n+1}{k} \binom{2n-k}{k-1} \right] 5^{n-k}. \tag{6}$$

Proof. It is sufficient to put $x = \varphi$ and $y = \psi$ in (3). □

In the sequel, we obtain some properties of $C_{m,n}(s,t)$ and $\mathcal{I}_m(n)$, when $m = 5$.

Proposition 3.6. *Inside the* $5 \times n$ *table we have*

$$\mathcal{D}(s+2,1) = \mathcal{I}_5(s) \quad \textit{and} \quad \mathcal{D}(s+2,3) = 2\mathcal{I}_5(s) - 1$$

for all $1 \leqslant s \leqslant n$.

Proof. From the table in Example 5.1, it follows simply that $\mathcal{I}_5(s) = \mathcal{D}(s+2,1)$ for all $s \geqslant 1$. Also, from the table, it follows that

$$2\mathcal{D}(s+1,1) - \mathcal{D}(s+1,3) = 2\mathcal{D}(s,1) - \mathcal{D}(s,3)$$

for all $s \geqslant 1$, that is, $2\mathcal{D}(s,1) - \mathcal{D}(s,3)$ is constant. Since $2\mathcal{D}(1,1) - \mathcal{D}(1,3) = 1$, we get $2\mathcal{D}(s+2,1) - \mathcal{D}(s+2,3) = 1$, from which the result follows. □

Proposition 3.7. *Inside the* $5 \times n$ *table we have*

$$\mathcal{D}(s,1) \times \mathcal{D}(s+t,3) - \mathcal{D}(s,3) \times \mathcal{D}(s+t,1) = \sum_{i=s}^{s+t-1} \mathcal{D}(i,2)$$

for all $1 \leqslant s, t \leqslant n$.

Proof. From Proposition 3.6, we know that $\mathcal{D}(s,3) = 2\mathcal{D}(s,1) - 1$ for all $1 \leqslant s \leqslant n$. Then

$$\begin{aligned}
&\mathcal{D}(s,1)\mathcal{D}(s+t,3) - \mathcal{D}(s,3)\mathcal{D}(s+t,1) \\
=&\mathcal{D}(s,1)(2\mathcal{D}(s+t,1) - 1) - (2\mathcal{D}(s,1) - 1)\mathcal{D}(s+t,1) \\
=&2\mathcal{D}(s,1)\mathcal{D}(s+t,1) - \mathcal{D}(s,1) - 2\mathcal{D}(s,1)\mathcal{D}(s+t,1) + \mathcal{D}(s+t,1) \\
=&\mathcal{D}(s+t,1) - \mathcal{D}(s,1).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathcal{D}(s+t,1) - \mathcal{D}(s,1) &= \mathcal{D}(s+t-1,1) + \mathcal{D}(s+t-1,2) - \mathcal{D}(s,1) \\
&= \mathcal{D}(s+t-2,1) + \mathcal{D}(s+t-2,2) + \mathcal{D}(s+t-1,2) - \mathcal{D}(s,1) \\
&\vdots \\
&= \sum_{i=s}^{s+t-1} \mathcal{D}(i,2) + \mathcal{D}(s,1) - \mathcal{D}(s,1) \\
&= \sum_{i=s}^{s+t-1} \mathcal{D}(i,2),
\end{aligned}$$

from which the result follows. □

4 Further results about lattice paths by using Fibonacci and Pell-Lucas numbers

In this section, we obtain some relations and properties about lattice paths by the aid of Fibonacci and Pell-Lucas sequences.

Proposition 4.1. *Inside the* $4 \times n$ *table we have*

$$\mathcal{D}(s,1) = \mathcal{F}_{2s-1} \quad \textit{and} \quad \mathcal{D}(s,2) = \mathcal{F}_{2s}$$

for all $s \geqslant 1$. *As a result,*

$$\mathcal{D}(s,1) \times \mathcal{D}(s+t,2) - \mathcal{D}(s,2) \times \mathcal{D}(s+t,1) = \mathcal{D}(s,2).$$

for all $s, t \geqslant 1$.

Proof. Clearly $\mathcal{D}(1,1) = \mathcal{D}(1,2) = \mathcal{F}_1 = \mathcal{F}_2 = 1$. Now since

$$\mathcal{D}(s,1) = \mathcal{D}(s-1,1) + \mathcal{D}(s-1,2),$$
$$\mathcal{D}(s,2) = 2\mathcal{D}(s-1,2) + \mathcal{D}(s-1,1).$$

we may prove, by using induction that, $\mathcal{D}(s,1) = \mathcal{F}_{2s-1}$ and $\mathcal{D}(s,2) = \mathcal{F}_{2s}$ for all $s \geqslant 1$. The second claim follows from the fact that

$$\mathcal{F}_{2s-1}\mathcal{F}_{2s+2t} - \mathcal{F}_{2s}\mathcal{F}_{2s+2t-1} = \mathcal{F}_{2s}.$$

The proof is complete. □

Proposition 4.2. *Inside the* $4 \times n$ *table we have*

$$\mathcal{I}_4(2s+1) = \frac{1}{4}\mathcal{I}_4(s+1)^2 + \mathcal{D}(s,2)^2$$

for all $1 \leqslant s \leqslant n$.

Proof. Following Lemma 3.4 and Proposition 4.1, it is enough to show that

$$2\mathcal{F}_{4s+3} = \mathcal{F}_{2s+3}^2 + \mathcal{F}_{2s}^2.$$

First observe that the equation $\mathcal{F}_{2n-1} = \mathcal{F}_n^2 + \mathcal{F}_{n-1}^2$ yields $\mathcal{F}_{4s+1} = \mathcal{F}_{2s+1}^2 + \mathcal{F}_{2s+2}^2$ and $\mathcal{F}_{4s+5} = \mathcal{F}_{2s+3}^2 + \mathcal{F}_{2s+2}^2$. Now, by combining these two formulas, we obtain

$$\begin{aligned}
\mathcal{F}_{2s+3}^2 + \mathcal{F}_{2s}^2 &= \mathcal{F}_{4s+5} + \mathcal{F}_{4s+1} - (\mathcal{F}_{2s+1}^2 + \mathcal{F}_{2s+2}^2) \\
&= \mathcal{F}_{4s+4} + \mathcal{F}_{4s+3} + \mathcal{F}_{4s+1} - \mathcal{F}_{4s+3} \\
&= \mathcal{F}_{4s+3} + \mathcal{F}_{4s+2} + \mathcal{F}_{4s+1} \\
&= 2\mathcal{F}_{4s+3},
\end{aligned}$$

as required. □

Pell numbers $\mathcal{P}_n$ are defined recursively as $\mathcal{P}_1 = 1$, $\mathcal{P}_2 = 2$, and $\mathcal{P}_n = 2\mathcal{P}_{n-1} + \mathcal{P}_{n-2}$ for all $n \geqslant 3$. The Binet's formula corresponding to $\mathcal{P}_n$ is $\mathcal{P}_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$.

Proposition 4.3. *Inside the* $3 \times n$ *table we have*

$$\mathcal{D}(s,1) = \mathcal{P}_s \quad \textit{and} \quad \mathcal{D}(s,2) = \mathcal{Q}_s$$

for all $s \geqslant 1$. *As a result,*

$$\mathcal{D}(s,1) \times \mathcal{D}(s+t,2) - \mathcal{D}(s,2) \times \mathcal{D}(s+t,1) = (-1)^{s+1}\mathcal{D}(t,1).$$

for all $s, t \geqslant 1$.

Proof. From the table in Lemma 3.2, we observe that

$$\mathcal{D}(s,1) = 2\mathcal{D}(s-1,1) + \mathcal{D}(s-2,1),$$
$$\mathcal{D}(s,2) = 2\mathcal{D}(s-1,2) + \mathcal{D}(s-2,2)$$

for all $s \geqslant 3$. Now since $\mathcal{D}(1,1) = \mathcal{P}_1 = 1$, $\mathcal{D}(2,1) = \mathcal{P}_2 = 2$, $\mathcal{D}(1,2) = \mathcal{Q}_1 = 1$, and $\mathcal{D}(2,2) = \mathcal{Q}_2 = 3$ one can show, by using induction, that $\mathcal{D}(s,1) = \mathcal{P}_s$ and $\mathcal{D}(s,2) = \mathcal{Q}_s$ for all s. To prove the second claim, we use the following formula

$$\mathcal{P}_s\mathcal{Q}_{s+t} - \mathcal{Q}_s\mathcal{P}_{s+t} = (-1)^{s+1}\mathcal{P}_t$$

that can be proved simply by using Binet's formulas. □

5 Further work

We end our paper with posing few open problems on determinant of matrices arising from lattice paths.

First consider the $m \times n$ table T with $2n \geqslant m$. For positive integers $\ell_1, \ell_2, \ldots, \ell_{\lceil \frac{m}{2} \rceil}$, we can write $\mathcal{I}_m(n)$ as

$$\mathcal{I}_m(n) = \ell_1 \mathcal{I}_m(n-1) + \ell_2 \mathcal{I}_m(n-2) + \cdots + \ell_{\lceil \frac{m}{2} \rceil} \mathcal{I}_m(n - \lceil \frac{m}{2} \rceil).$$

Also, for positive integers $0 \leqslant s \leqslant \lceil \frac{m}{2} \rceil$ and $k_{1,s}, k_{2,s}, \ldots, k_{\lceil \frac{m}{2} \rceil, s}$, we put

$$\mathcal{I}_m(n-s) = k_{1,s}x_1 + k_{2,s}x_2 + \cdots + k_{\lceil \frac{m}{2} \rceil, s}x_{\lceil \frac{m}{2} \rceil},$$

where $x_t = \mathcal{D}(n - \lceil \frac{m}{2} \rceil, t) = \sum_{i=1}^{m} \mathcal{D}^i(n - \lceil \frac{m}{2} \rceil, t)$ is the number of all lattice paths from the first column to the $(n - \lceil \frac{m}{2} \rceil, t)$-cell of T, for each $1 \leqslant i \leqslant m$ and $1 \leqslant t \leqslant \lceil \frac{m}{2} \rceil$. Utilizing the above notation, we can can write

$$\begin{aligned}
\mathcal{I}_m(n) =& k_{1,0}x_1 + k_{2,0}x_2 + \cdots + k_{\lceil \frac{m}{2} \rceil, 0}x_{\lceil \frac{m}{2} \rceil} \\
=& \ell_1 \mathcal{I}_{n-1} + \ell_2 \mathcal{I}_{n-2} + \cdots + \ell_{\lceil \frac{m}{2} \rceil} \mathcal{I}_{n - \lceil \frac{m}{2} \rceil} \\
=& \ell_1 (k_{1,1}x_1 + k_{2,1}x_2 + \cdots + k_{\lceil \frac{m}{2} \rceil, 1}x_{\lceil \frac{m}{2} \rceil}) \\
& + \ell_2 (k_{1,2}x_1 + k_{2,2}x_2 + \cdots + k_{\lceil \frac{m}{2} \rceil, 2}x_{\lceil \frac{m}{2} \rceil}) \\
& \vdots \\
& + \ell_{\lceil \frac{m}{2} \rceil} (k_{1,\lceil \frac{m}{2} \rceil}x_1 + k_{2,\lfloor \frac{m}{2} \rfloor}x_2 + \cdots + k_{\lceil \frac{m}{2} \rceil, \lceil \frac{m}{2} \rceil}x_{\lceil \frac{m}{2} \rceil}).
\end{aligned} \tag{7}$$

From (7), we obtain the following system of linear equations

$$\begin{cases}
k_{1,1}\ell_1 & + & \cdots & + & k_{1,\lceil \frac{m}{2} \rceil}\ell_{\lceil \frac{m}{2} \rceil} & = & k_{1,0}, \\
k_{2,1}\ell_1 & + & \cdots & + & k_{2,\lceil \frac{m}{2} \rceil}\ell_{\lceil \frac{m}{2} \rceil} & = & k_{2,0}, \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
k_{\lceil \frac{m}{2} \rceil,1}\ell_1 & + & \cdots & + & k_{\lceil \frac{m}{2} \rceil,\lceil \frac{m}{2} \rceil}\ell_{\lceil \frac{m}{2} \rceil} & = & k_{\lceil \frac{m}{2} \rceil,0}.
\end{cases} \tag{8}$$

Now consider the following coefficient matrix A of the system (8)

$$A = \begin{bmatrix}
k_{1,1} & k_{1,2} & \cdots & k_{1,\lceil \frac{m}{2} \rceil} \\
k_{2,1} & k_{2,2} & \cdots & k_{2,\lceil \frac{m}{2} \rceil} \\
\vdots & \vdots & \ddots & \vdots \\
k_{\lceil \frac{m}{2} \rceil,1} & k_{\lceil \frac{m}{2} \rceil,2} & \cdots & k_{\lceil \frac{m}{2} \rceil,\lceil \frac{m}{2} \rceil}
\end{bmatrix},$$

which we call the *coefficient matrix* of the table T and denote it by $\mathcal{D}(T)$.

Conjecture 1. For a given $m \times n$ table T $(2n \geqslant m)$, we have $\det(\mathcal{D}(T)) = -2^{\lfloor \frac{m}{2} \rfloor}$.

Example 5.1. Let T be a $5 \times n$ table. The columns $n-3$, $n-2$, $n-1$, and n of T are given by

$n-3$	$n-2$	$n-1$	n
x_1	$x_1 + x_2$	$2x_1 + 2x_2 + x_3$	$4x_1 + 6x_2 + 3x_3$
x_2	$x_1 + x_2 + x_3$	$2x_1 + 4x_2 + 2x_3$	$6x_1 + 10x_2 + 6x_3$
x_3	$2x_2 + x_3$	$2x_1 + 4x_2 + 3x_3$	$6x_1 + 12x_2 + 7x_3$
x_2	$x_1 + x_2 + x_3$	$2x_1 + 4x_2 + 2x_3$	$6x_1 + 10x_2 + 6x_3$
x_1	$x_1 + x_2$	$2x_1 + 2x_2 + x_3$	$4x_1 + 6x_2 + 3x_3$

from which it follows that

$$\begin{aligned}
\mathcal{I}_5(n-3) &= 2x_1 + 2x_2 + x_3, \\
\mathcal{I}_5(n-2) &= 4x_1 + 6x_2 + 3x_3, \\
\mathcal{I}_5(n-1) &= 10x_1 + 16x_2 + 9x_3, \\
\mathcal{I}_5(n) &= 28x_1 + 44x_2 + 25x_3
\end{aligned}$$

Clearly,

$$\mathcal{I}_5(n) = \ell_1\mathcal{I}_5(n-1) + \ell_2\mathcal{I}_5(n-2) + \ell_3\mathcal{I}_5(n-3)$$

for some ℓ_1, ℓ_2, ℓ_3, and that the coefficient matrix of the table T is $\mathcal{D}(T) = \begin{bmatrix} 10 & 4 & 2 \\ 16 & 6 & 2 \\ 9 & 3 & 1 \end{bmatrix}$. It is obvious that $\det(\mathcal{D}(T)) = -2^{\lfloor \frac{5}{2} \rfloor} = -4$.

Our second problem is to compute the determinant of special *Hankel matrices*. Recall that a Hankel matrix (or catalecticant matrix) of a numerical sequence $\mathcal{D} = \{c_i\}$, named after Hermann Hankel, is a matrix defined as

$$H_n^t(\mathcal{D}) = \begin{bmatrix} c_t & c_{t+1} & c_{t+2} & \cdots & c_{t+n-1} \\ c_{t+1} & c_{t+2} & c_{t+3} & \cdots & c_{t+n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{t+n-1} & c_{t+n} & c_{t+n+1} & \cdots & c_{t+2n-2} \end{bmatrix}.$$

In [32, Theorems 3 and 4], the authors use a sequence of ideas to reduce the problem to a previous work of Cigler and Krattenthaler [4] (the first paper of this series), which describes the Hankel determinants $\det H_n^1(\mathcal{D})$ and $\det H_n^2(\mathcal{D})$ of some similar sequences $\mathcal{D}$. Now, consider the sequence $\mathcal{D}$ with elements $1, 1, 2, 5, 13, 35, 96, \ldots$ (see A005773). In the following, we suggest the values of the determinant of the Hankel matrix $H_n^0(\mathcal{D})$

Conjecture 2. For positive integers n, consider the Hankel matrix

$$H_n^0(\mathcal{D}) = \begin{bmatrix} 1 & 1 & 2 & 5 & \cdots & c_n \\ 1 & 2 & 5 & 13 & \cdots & c_{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ c_n & c_{n+1} & c_{n+2} & c_{n+3} & \cdots & c_{2n} \end{bmatrix}.$$

Then

$$\det H_n^0(\mathcal{D}) = \begin{cases} 0, & n \equiv 3 \pmod 6, \\ -1, & n \equiv 4, 5 \pmod 6, \\ 1, & n \equiv 2, 3 \pmod 6. \end{cases}$$

Problem 1. How can we compute $\det(H_n^t)$?

Conjecture 3. We say that a matrix D is totally positive if all its minors are non-negative. The Riordan array matrix of $D(n, n)$ is totally positive.

Problem 2. Find combinatorial bijection for Somos identity in 2.2.

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