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On k-dprime Divisor Function Graph

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Abstract

Let p and q be distinct primes. The *semiprime divisor function graph* denoted by $G_{D(pq)}$, is the graph with vertex set $V(G_{D(pq)}) = \{1, p, q, pq\}$ and edge set $E(G_{D(pq)}) = \{\{1, p\}, \{1, q\}, \{1, pq\}, \{p, pq\}, \{q, pq\}\}$. The semiprime divisor function graph is a special type of divisor function graph $G_{D(n)}$ in which $n = pq$. Recently, the energy and some indices of semiprime divisor function graph have been determined. In this paper, we introduce a natural extension to the semiprime divisor function graph which we call the *k-dprime divisor function graph.* Moreover, we present results on some distance-based and degree-based topological indices of k-dprime divisor function graph. We end the paper by giving some open problems.

Keywords: k-dprime divisor function graph, semiprime divisor function graph, divisor function graph, topological indices

Mathematics Subject Classification [2010]: 05C12, 05C50, 05C85

1 Introduction

One of the developing areas in Graph theory is the notion of using Number theory concepts to define graphs. The said graphs are called *number theoretic based graphs.* One of the most studied number theoretic based graph is the *divisor graph.* Let S be a non-empty subset of $\mathbb{Z}$, a graph $G(V, E)$ is a **divisor graph** if $V(G) = S$ and $E(G) = \{ij : \text{either } i \mid j \text{ or } j \mid i \text{ for } i, j \in V(G) \text{ with } i \neq j\}$. The concept of divisor graph was introduced by Singh and Santhosh [1] in 2000. Since then, various research studies about divisor graph have been conducted (see [2, 3, 4, 5]).

Motivated by the concept of divisor graph, Kannan et al. [6] introduced the concept of *divisor function graph* in 2015. Let $n \geq 1$ be an integer, and suppose that n has r divisors $d_1, d_2, \ldots, d_r$, the **divisor function graph** of n denoted by $G_{D(n)}(V, E)$ is the graph with $V(G_{D(n)}) = \{d_1, d_2, \ldots, d_r\}$ and

$$E(G_{D(n)}) = \{d_i d_j : \text{either } d_i \mid d_j \text{ or } d_j \mid d_i \text{ for } d_i, d_j \in V(G)_{D(n)} \text{ with } i \neq j\}.$$

If in the definition of the divisor function graph we have $n = pq$, for distinct primes p and q, then $G_{D(n)}$ is called a **semiprime divisor function graph**. The concept of semiprime divisor function graph was introduced recently by Shanmugavelan et al. in [7]. Also in [7], Shanmugavelan et al. determined the energy and some indices of the semiprime divisor function graph.

Inspired by the work of Shanmugavelan et al., we introduce a natural extension to the semiprime divisor function graph which we call the *k-dprime divisor function graph* in this paper. We then determine some distance-based and degree-based topological indices of the k-dprime divisor function graph. We also give some problems that the reader might consider as a research study.

[1]speaker

2 The k-dprime Divisor Function Graph

Unless otherwise stated, we follow the graph theory notations of Bondy and Murty [8] and the number theory notations of Burton [9]. We now formally define the k-dprime divisor function graph.

Definition 2.1. Let $n \geq 1$ be an integer such that $n = p_1p_2 \dots p_k$, where each p_i are distinct primes for $i = 1, 2, \dots, k$. The graph $G_{D(n)}(V, E)$ with $V(G_{D(n)}) = \{u : u \mid n\}$ and

$$E(G_{D(n)}) = \{uv : \text{either } u \mid v \text{ or } v \mid u \text{ for } u, v \in V(G_{D(n)}) \text{ with } u \neq v\}$$

is called a ***k-dprime divisor function graph.***

Example 2.2. The graph of 3-drpime and 4-dprime divisor function graph is given in Figure 1. On the other hand, the graph of 5-dprime divisor function graph is given in Figure 2.

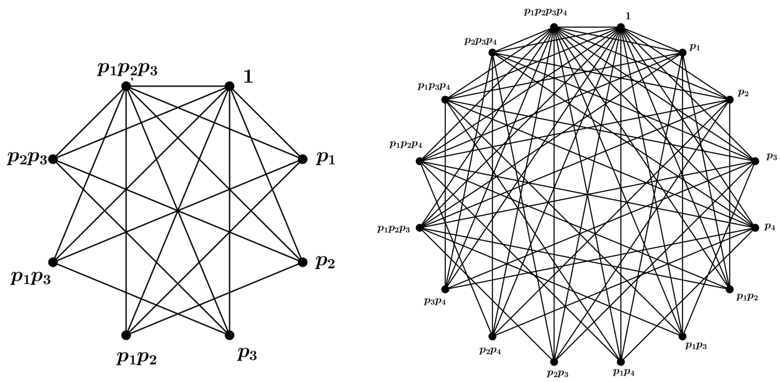

Figure 1: The 3-dprime and 4-dprime divisor function graph.

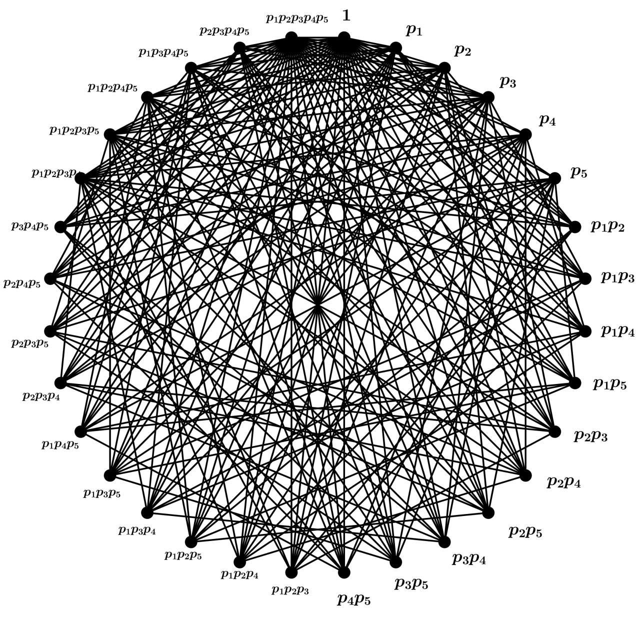

Figure 2: The 5-dprime divisor function graph.

Observe that the number of vertices in 3-dprime, 4-dprime, and 5-dprime divisor function graph are 8, 16, and 32, respectively. Moreover, the degree sequence of the vertices in 3-dprime, 4-dprime, and 5-dprime divisor function graph are $(7, 4, 4, 4, 4, 4, 4, 7)$, $(15, 8, 8, 8, 8, 6, 6, 6, 6, 6, 6, 8, 8, 8, 8, 15)$, and $(31, 16, 16, 16, 16, 16, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 10, 16, 16, 16, 16, 16, 31)$, respectively. Finally, the number of edges in 3-dprime, 4-dprime, and 5-dprime divisor function graph are 19, 65, and 211, respectively.

For simplicity, we just denote by Γ_k the k- dprime divisor function graph $G_{D(n=p_1p_2\dots p_k)}$. Some of the basic properties of Γ_k is given in the next series of results.

Theorem 2.3. *The number of vertices in Γ_k is 2^k, that is, the order of Γ_k is 2^k.*

Proof. The result follows from the composition of vertices in Γ_k and the fact that an integer with canonical representation $p_1^{e_1}p_2^{e_2}\dots p_r^{e_r}$ has $(e_1+1)\cdot(e_2+1)\cdot\ldots\cdot(e_r+1)$ divisors. $\square$

Remark 2.4. The number of vertices in 3-dprime, 4-dprime, and 5-dprime divisor function graph that were obtained in Example 2.2 agrees with Theorem 2.3.

Theorem 2.5. *If $v \in V(\Gamma_k)$, then*

$$deg_{\Gamma_k}(v) = \begin{cases} 2^k - 1 & \text{if } v = 1, n \\ 2^{\omega(v)} + 2^{k-\omega(v)} - 2 & \text{otherwise} \end{cases}$$

where $\omega(v)$ is the number of distinct prime divisors of v.

Proof. To prove the theorem, we will consider 3 cases.

Case 1: If $v = 1$. If $v = 1$, then there are 2^k integers in $V(\Gamma_k)$ (including v) that are divisible by v. By noting that in $E(\Gamma_k)$ we must have $u \neq v$, we conclude that there are $2^k - 1$ vertices that are incident to v, by considering the number of integers in $V(\Gamma_k)$ that are divisible by v. Next, we consider the number of integers in $V(\Gamma_k)$ that divides v. The only integer in $V(\Gamma_k)$ that divides v is v itself. By noting that in $E(\Gamma_k)$ we must have $u \neq v$, we conclude that there is no vertex incident to v, by considering the number of integers in $V(\Gamma_k)$ that divides v. All in all, we have $2^k - 1$ edges incident to v. Hence, $deg_{\Gamma_k}(v) = 2^k - 1$.

Case 2: If $v = n$. If $v = n$, then there is one integer in $V(\Gamma_k)$ that is divisible by v, v itself. By noting that in $E(\Gamma_k)$ we must have $u \neq v$, we conclude that there is no vertex incident to v, by considering the number of integers in $V(\Gamma_k)$ that are divisible by v. Next, we consider the number of integers in $V(\Gamma_k)$ that divides v. There are 2^k integers in $V(\Gamma_k)$ (including v) that divides v. By noting that in $E(\Gamma_k)$ we must have $u \neq v$, we conclude that there are $2^k - 1$ vertices incident to v, by considering the number of integers in $V(\Gamma_k)$ that divides v. All in all, we have $2^k - 1$ edges incident to v. Hence, $deg_{\Gamma_k}(v) = 2^k - 1$.

Case 3: If $v \in V(\Gamma_k) - \{1, n\}$. Let $v \in V(\Gamma_k) - \{1, n\}$ and denote by $\omega(v)$ the number of its distinct prime divisors. Note that since $v \mid n$, and $n = p_1p_2\dots p_k$, we know that v is a product of $\omega(v)$ distinct primes.

Now, let us first count the number of integers in $V(\Gamma_k)$ that divides v. Since v is a product of $\omega(v)$ distinct primes, if we use the fact that an integer with canonical representation $p_1^{e_1}p_2^{e_2}\dots p_r^{e_r}$ has $(e_1+1)\cdot(e_2+1)\cdot\ldots\cdot(e_r+1)$ divisors, we conclude that there are $2^{\omega(v)}$ integers in $V(\Gamma_k)$ that divides v. But in $E(\Gamma_k)$ we must have $u \neq v$, so if we consider the number of integers in $V(\Gamma_k)$ that divides v, we have $2^{\omega(v)} - 1$ edges that are incident to v. Before we proceed, we note that the vertices u that are incident to v in this case satisfies the inequality $\omega(u) < \omega(v)$. This is because (i) $u \leq v$ and (ii) if $u \in V(\Gamma_k)$ such that $u \neq v$, and $\omega(u) = \omega(v)$ then $u \nmid v$.

Next, we count the number of integers in $V(\Gamma_k)$ that are divisible by v. Note that if $u \in V(\Gamma_k)$ such that $u \neq v$, and $\omega(u) = \omega(v)$ then $v \nmid u$. So we start the counting for integers u with $\omega(u) > \omega(v)$. By **counting**, there are $\binom{k-\omega(v)}{1}$ integers u in $V(\Gamma_k)$ with $\omega(u) = \omega(v) + 1$. Similarly, by counting, we know that there are $\binom{k-\omega(v)}{2}$ integers u in $V(\Gamma_k)$ with $\omega(u) = \omega(v) + 2$. In general, for $j = 1, 2, \dots, k - \omega(v)$, a counting technique asserts that there are $\binom{k-\omega(v)}{j}$ integers u in $V(\Gamma_k)$ with $\omega(u) = \omega(v) + j$. All in all we have $\displaystyle\sum_{j=1}^{k-\omega(v)} \binom{k-\omega(v)}{j}$ number of integers in $V(\Gamma_k)$ that are divisible by v that are not equal to v.

Hence, there are $2^{\omega(v)} - 1 + \displaystyle\sum_{j=1}^{k-\omega(v)} \binom{k-\omega(v)}{j}$ number of vertices that are incident to v. If we use the identity $\displaystyle\sum_{j=0}^{n} \binom{n}{j} = 2^n$, we conclude that there are $2^{\omega(v)} + 2^{k-\omega(v)} - 2$ incident edges to v. Hence, the degree of v is given by $2^{\omega(v)} + 2^{k-\omega(v)} - 2$.

□

Remark 2.6. The degree of a vertex in 3-dprime, 4-dprime, and 5-dprime divisor function graph that were obtained in Example 2.2 agrees with Theorem 2.5.

Remark 2.7. Let $j = 0, 1, \ldots, k$. In Γ_k, there are $\binom{k}{j}$ vertices with j distinct prime divisors.

The next result gives a recursive formula in determining the size of Γ_k. The formula is dependent on the size of Γ_{k-1} and the degree of vertices in Γ_{k-1}.

Lemma 2.8. *Let Γ_k and Γ_{k-1} denote the k-dprime and $k-1$-dprime divisor function graph respectively. If Γ_{k-1} has degree sequence $(deg(v_1), deg(v_2), \ldots, deg(v_{2^{k-1}}))$ arranged in increasing order of number of distinct divisors, then*

$$|E(\Gamma_k)| = |E(\Gamma_{k-1})| + \sum_{i=1}^{2^{k-1}} (deg(v_i) + 1).$$

Proof. First, note that Γ_{k-1} is a subgraph of Γ_k. So, all the edges in Γ_{k-1} also belong to Γ_k. Also, observe that $V(\Gamma_k) = V(\Gamma_{k-1}) \cup \{vp_k : v \in V(\Gamma_{k-1})\}$. This means that in order to determine the number of edges of Γ_k, it is enough to consider the number of edges contributed by the vertices in $\{vp_k : v \in V(\Gamma_{k-1})\}$ and add it to $|E(\Gamma_{k-1})|$.

We claim that if $u \in \{vp_k : v \in V(\Gamma_{k-1})\}$ then $u = vp_k$ contributes $deg(v) + 1$ edges in the graph Γ_k. To prove our claim, we proceed by counting the number of edges contributed by u in Γ_k avoiding duplication, which is equal to the number of integers in $V(\Gamma_{k-1})$ that divides u added by the number of integers in $\{vp_k : v \in V(\Gamma_{k-1})\}$ that are divisible by u.

The number of integers in $V(\Gamma_{k-1})$ dividing u is equal to the number of integers in $V(\Gamma_{k-1})$ that divides v plus one (since $v \mid u$). So, we have $2^{\omega(v)}$ integers dividing u in $V(\Gamma_{k-1})$. On the other hand, the number of integers in $\{vp_k : v \in V(\Gamma_{k-1})\}$ that are divisible by u is equal to the number of integers divisible by v in $V(\Gamma_{k-1})$ which is $2^{(k-1)-\omega(v)} - 1$. All in all, u contributes a total of $2^{\omega(v)} + 2^{(k-1)-\omega(v)} - 1 = deg(v) + 1$.

Using the just proved claim, we conclude that there are a total of $\sum_{i=1}^{2^{k-1}} (deg(v_i) + 1)$ edges contributed by the vertices in the set $\{vp_k : v \in V(\Gamma_{k-1})\}$ in the graph Γ_k. If we add that sum to $|E(\Gamma_{k-1})|$ we have $|E(\Gamma_k)|$. □

A formula on how to compute for $|E(\Gamma_k)|$ using only the variable k is given in the next theorem.

Theorem 2.9. *The graph Γ_k has $3^k - 2^k$ number of edges. That is, the size of Γ_k is $3^k - 2^k$.*

Proof. If we combine Lemma 2.8 with Theorem 2.5 and Remark 2.7 we get

$$|E(\Gamma_k)| = |E(\Gamma_{k-1})| + \sum_{j=0}^{k-1} \binom{k-1}{j} (2^{k-j-1} + 2^j - 1).$$

Now, we wish to simplify the expression $\sum_{j=0}^{k-1} \binom{k-1}{j} (2^{k-j-1} + 2^j - 1)$ in the above equation by using the identities $\sum_{j=0}^{n} \binom{n}{j} = 2^n$ and $\sum_{j=0}^{n} \binom{n}{j} 2^j = 3^n$. By simplifying, we have

$$\begin{aligned}
\sum_{j=0}^{k-1} \binom{k-1}{j} (2^{k-j-1} + 2^j - 1) &= \sum_{j=0}^{k-1} \binom{k-1}{j} (2^{k-1-j}) + \sum_{j=0}^{k-1} \binom{k-1}{j} (2^j) - \sum_{j=0}^{k-1} \binom{k-1}{j} \\
&= 3^{k-1} + 3^{k-1} - 2^{k-1} \\
&= 2(3^{k-1}) - 2^{k-1}.
\end{aligned}$$

Thus, we now have

$$|E(\Gamma_k)| = |E(\Gamma_{k-1})| + 2(3^{k-1}) - 2^{k-1}. \tag{1}$$

We note that the 0-dprime divisor function graph has $|E(\Gamma_0)| = 0$. So, solving the recurrence relation in Equation (1) with the initial condition $|E(\Gamma_0)| = 0$ gives

$$\begin{aligned}|E(\Gamma_k)| &= |E(\Gamma_0)| + 2\sum_{j=0}^{k-1} 3^j - \sum_{j=0}^{k-1} 2^j \\ &= |E(\Gamma_0)| + 2\left(\frac{3^k - 1}{2}\right) - (2^k - 1) \\ &= 0 + 3^k - 1 - 2^k + 1 \\ &= 3^k - 2^k.\end{aligned}$$

□

Remark 2.10. Using Theorem 2.9, one can verify that Γ_3 has $3^3 - 2^3 = 19$ number of edges. Also, Γ_4 has $3^4 - 2^4 = 65$ number of edges. Finally, Γ_5 has $3^5 - 2^5 = 211$ number of edges. The results agree with the results stated in Example 2.2.

We now end the section by presenting some results about distance between vertices in k-dprime divisor function graph.

Lemma 2.11. *Let Γ_k denote the k-dprime divisor function graph. If $u, v \in V(\Gamma_k)$ then*

$$d_{\Gamma_k}(u,v) = \begin{cases} 0 & \text{if } u = v \\ 1 & \text{if } u \text{ is adjacent to } v \\ 2 & \text{otherwise.} \end{cases}$$

Proof. Clearly, $d_{\Gamma_k}(u,v) = 0$ if $u = v$ and $d_{\Gamma_k}(u,v) = 1$ if u is adjacent to v. Now, if u is not adjacent to v, then the path $u \to 1 \to v$ is a shortest path from u to v. Another shortest path from u to v is the path $u \to n \to v$. Hence, $d_{\Gamma_k}(u,v) = 2$, if u is not adjacent to v. □

Corollary 2.12. *Let Γ_k denote the k-dprime divisor function graph. The diameter of Γ_k denoted by $diam(\Gamma_k)$ is 2.*

3 Some Indices of the k-dprime Divisor Function Graph

Given a family of graphs $\mathcal{G}$, a **topological index** is a function $Top : \mathcal{G} \to \mathbb{R}$ such that if $\Gamma_1, \Gamma_2 \in \mathcal{G}$, and $\Gamma_1 \cong \Gamma_2$ then $Top(\Gamma_1) = Top(\Gamma_2)$. In this section, we give some general results about the following distance-based and degree-based topological indices of the k-dprime divisor function graph Γ_k

Wiener Index: $$W(\Gamma_k) = \sum_{\{u,v\}\subseteq V(\Gamma_k)} d_{\Gamma_k}(u,v)$$

Hyper-Wiener Index: $$WW(\Gamma_k) = \frac{1}{2}\sum_{\{u,v\}\subseteq V(\Gamma_k)} [d_{\Gamma_k}(u,v) + (d_{\Gamma_k}(u,v))^2]$$

Harary Index: $$H(\Gamma_k) = \sum_{\{u,v\}\subseteq V(\Gamma_k)} \frac{1}{d_{\Gamma_k}(u,v)}$$

$$\textbf{First Zagreb Index: } M_1(\Gamma_k) = M_1(\Gamma_k) = \sum_{u \in V(\Gamma_k)} (deg_{\Gamma_k}(u))^2.$$

To effectively calculate the first three indices, we need to recall the concept of graph's *distance matrix* as well as its variants, the *square distance matrix* and the *reciprocal distance matrix.* The **distance matrix** of a graph G of order $|V(G)|$, denoted by $\mathbf{D}(G)$ is the $|V(G)| \times |V(G)|$ matrix $\mathbf{D}$ with entries $[d_{ij}] = d_G(v_i, v_j)$. On the other hand, the **square distance matrix** of a graph G, denoted by $\mathbf{D}^2(G)$ is the matrix with ij-entry equal to $(d_G(v_i, v_j))^2$. Lastly, the **reciprocal distance matrix** of a graph G, denoted by $\mathbf{D}^{-1}(G)$ is the matrix with ij-entry equal to $\frac{1}{d_G(v_i,v_j)}$. Once the distance matrix of a graph and its variants have been determined, the Wiener, hyper-Wiener, and Harary index of the gaph can be easily calculated as shown in the next example.

Example 3.1. It follows from the graph of the 3-dprime divisor function graph Γ_3 in Example 2.2 that

$$\mathbf{D}(\Gamma_3) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 2 & 1 & 1 & 2 & 1 \\ 1 & 2 & 0 & 2 & 1 & 2 & 1 & 1 \\ 1 & 2 & 2 & 0 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 2 & 2 & 1 \\ 1 & 1 & 2 & 1 & 2 & 0 & 2 & 1 \\ 1 & 2 & 1 & 1 & 2 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

where the matrix is indexed by the ordered set $\{1, p_1, p_2, p_3, p_1p_2, p_1p_3, p_2p_3, p_1p_2p_3\}$. If we use the definition of the Wiener index, we have

$$\begin{aligned} W(\Gamma_3) &= \sum_{\{u,v\} \subseteq V(\Gamma_3)} d_{\Gamma_3}(u, v) \\ &= \sum_{v \in V(\Gamma_3)} d_{\Gamma_3}(1, v) + \sum_{v \in V(\Gamma_3) - \{1\}} d_{\Gamma_3}(p_1, v) + \ldots + \sum_{v \in V(\Gamma_3) - \{1, p_1, p_2, \ldots, p_2p_3\}} d_{\Gamma_3}(p_1p_2p_3, v) \\ &= \frac{1}{2} \sum_{1 \le i,j \le |V(\Gamma_3)|} [d_{ij}] \\ &= \frac{74}{2} \\ &= 37. \end{aligned}$$

In a similar manner, one can show that

$$\sum_{\{u,v\} \subseteq V(\Gamma_3)} (d_{\Gamma_3}(u, v))^2 = \frac{1}{2} \sum_{1 \le i,j \le |V(\Gamma_3)|} [d_{ij}]^2$$

and

$$\sum_{\{u,v\} \subseteq V(\Gamma_3)} \frac{1}{d_{\Gamma_3}(u, v)} = \frac{1}{2} \sum_{1 \le i,j \le |V(\Gamma_3)|} \frac{1}{[d_{ij}]}.$$

So, by knowing the matrices

$$\mathbf{D}^2(\Gamma_3) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 4 & 4 & 1 & 1 & 4 & 1 \\ 1 & 4 & 0 & 4 & 1 & 4 & 1 & 1 \\ 1 & 4 & 4 & 0 & 4 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4 & 0 & 4 & 4 & 1 \\ 1 & 1 & 4 & 1 & 4 & 0 & 4 & 1 \\ 1 & 4 & 1 & 1 & 4 & 4 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

and

$$\mathbf{D}^{-1}(\Gamma_3) = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1/2 & 1/2 & 1 & 1 & 1/2 & 1 \\ 1 & 1/2 & 0 & 1/2 & 1 & 1/2 & 1 & 1 \\ 1 & 1/2 & 1/2 & 0 & 1/2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1/2 & 0 & 1/2 & 1/2 & 1 \\ 1 & 1 & 1/2 & 1 & 1/2 & 0 & 1/2 & 1 \\ 1 & 1/2 & 1 & 1 & 1/2 & 1/2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix},$$

we can easily compute the hyper-Wiener index and the Harary index of the 3-dprime divisor function graph as shown in the next page.

$$\begin{aligned} WW(\Gamma_3) &= \frac{1}{2} \sum_{\{u,v\} \subseteq V(\Gamma_3)} [d_{\Gamma_3}(u,v) + (d_{\Gamma_3}(u,v))^2] \\ &= \frac{1}{2} \left[\sum_{\{u,v\} \subseteq V(\Gamma_3)} d_{\Gamma_3}(u,v) + \sum_{\{u,v\} \subseteq V(\Gamma_3)} (d_{\Gamma_3}(u,v))^2 \right] \\ &= \frac{1}{2} \left[W(\Gamma_3) + \frac{1}{2} \sum_{1 \leq i,j \leq |V(\Gamma_3)|} [d_{ij}]^2 \right] \\ &= \frac{1}{2}(37 + 55) \\ &= 46. \end{aligned}$$

$$\begin{aligned} H(\Gamma_3) &= \sum_{\{u,v\} \subseteq V(\Gamma_3)} \frac{1}{d_{\Gamma_3}(u,v)} \\ &= \frac{1}{2} \sum_{1 \leq i,j \leq |V(\Gamma_3)|} \frac{1}{[d_{ij}]} \\ &= \frac{1}{2}(47) \\ &= 23.5. \end{aligned}$$

Remark 3.2. In general, given a connected graph G, the value of $W(G)$ can be computed by adding all the entries in $\mathbf{D}(G)$ and then dividing the result by 2. For the Harary index, it can be computed by adding all the entries in $\mathbf{D}^{-1}(G)$ and then dividing the result by 2. Finally, for the hyper-Wiener index, it can be calculated by adding half of $W(G)$ to quarter of the sum of all the entries in $\mathbf{D}^2(G)$.
Before we present the general results, we emphasize that the sum of all the vertex degrees in a graph G is equal to $2|E(G)|$. Now, we present the general results.

Theorem 3.3. *Let Γ_k denote the k-dprime divisor function graph. The Wiener index of Γ_k, denoted by $W(\Gamma_k)$ is given by*

$$W(\Gamma_k) = 2^{2k} - 3^k.$$

Proof. Let us denote by $S(\mathbf{D}(\Gamma_k))$, the sum of all the entries in $\mathbf{D}(\Gamma_k)$. Note that the entries in $\mathbf{D}(\Gamma_k)$ are either 0, 1, and 2 as stated in Lemma 2.11, and that all in all we have $2^k \times 2^k = 2^{2k}$ entries. Clearly, there are 2^k 0's in $\mathbf{D}(\Gamma_k)$. On the other hand, there are $2(3^k - 2^k)$ entries in $\mathbf{D}(\Gamma_k)$ whose value is 1. This is because $d_{\Gamma_k}(u, v) = 1$ implies u and v are adjacent, which contributes one count in the vertex degree of u and v respectively. Hence, the total number of 1's in $\mathbf{D}(\Gamma_k)$ corresponds to the sum of all vertex degrees in Γ_k which is equal to $2|E(\Gamma_k)|$. If we apply Theorem 2.9 we get the result that there are $2(3^k - 2^k)$ entries in $\mathbf{D}(\Gamma_k)$ whose value is 1. Finally, there are $2^{2k} - 2^k - 2(3^k - 2^k)$ entries in $\mathbf{D}(\Gamma_k)$ with value 2.

Now, by Remark 3.2 we have

$$\begin{aligned}
W(\Gamma_k) &= \frac{S(\mathbf{D}(\Gamma_k))}{2} \\
&= \frac{2^k(0) + 2(3^k - 2^k)(1) + [2^{2k} - 2^k - 2(3^k - 2^k)](2)}{2} \\
&= (3^k - 2^k) + 2^{2k} - 2^k - 2(3^k - 2^k) \\
&= 2^{2k} - 2^k - (3^k - 2^k) \\
&= 2^{2k} - 3^k.
\end{aligned}$$

□

Theorem 3.4. *Let Γ_k denote the k-dprime divisor function graph. The hyper-Wiener index of Γ_k, denoted by $WW(\Gamma_k)$ is given by*

$$WW(\Gamma_k) = 2^{k-1}(2^{k+1} + 2^k + 1) - 2(3^k).$$

Proof. If we use proof tecnique similar to the proof of Theorem 3.3 we have

$$\begin{aligned}
WW(\Gamma_k) &= \frac{1}{2}W(\Gamma_k) + \frac{S(\mathbf{D}^2(\Gamma_k))}{4} \\
&= \frac{1}{2}W(\Gamma_k) + \left[\frac{2^k(0) + 2(3^k - 2^k)(1) + [2^{2k} - 2^k - 2(3^k - 2^k)](4)}{4}\right] \\
&= \frac{1}{2}(2^{2k} - 3^k) + \left[\frac{2(3^k - 2^k) + 2^{2k+2} - 2^{k+2} - 8(3^k - 2^k)}{4}\right] \\
&= \frac{2^{2k+1} - 2(3^k) + 2^{2k+2} - 2^{k+2} - 6(3^k - 2^k)}{4} \\
&= \frac{2^k(2^{k+2} + 2^{k+1} + 2) - 8(3^k)}{4} \\
&= \frac{2^{k+1}(2^{k+1} + 2^k + 1) - 8(3^k)}{4} \\
&= 2^{k-1}(2^{k+1} + 2^k + 1) - 2(3^k).
\end{aligned}$$

□

Theorem 3.5. *Let Γ_k denote the k-dprime divisor function graph. The Harary index of Γ_k, denoted by $H(\Gamma_k)$ is given by*

$$H(\Gamma_k) = \frac{2^{k-1}(2^k - 3) + 3^k}{2}.$$

Proof. If we use proof tecnique similar to the proof of Theorem 3.3 we have

$$
\begin{aligned}
H(\Gamma_k) &= \frac{S(\mathbf{D}^{-1}(\Gamma_k))}{2} \\
&= \frac{2^k(0) + 2(3^k - 2^k)(1) + [2^{2k} - 2^k - 2(3^k - 2^k)]\left(\frac{1}{2}\right)}{2} \\
&= \frac{2(3^k - 2^k) + 2^{2k-1} - 2^{k-1} - (3^k - 2^k)}{2} \\
&= \frac{2^{2k-1} - 2^{k-1} + 3^k - 2^k}{2} \\
&= \frac{2^{k-1}(2^k - 3) + 3^k}{2}.
\end{aligned}
$$

□

Remark 3.6. If we use the results in Theorems 3.3, 3.4, and 3.5 to determine the Wiener, hyper-Wiener, and Harary index of Γ_3, we get $W(\Gamma_3) = 37$, $WW(\Gamma_3) = 46$, and $H(\Gamma_3) = 23.5$. The computed values agree with those presented in Example 3.1.

We now end the section by giving the general formula in determining the first Zagreb index of the k-dprime divisor function graph.

Theorem 3.7. *Let Γ_k denote the k-dprime divisor function graph. The first Zagreb Index of Γ_k, denoted by $M_1(\Gamma_k)$ is given by*

$$M_1(\Gamma_k) = 2(2^k - 1)^2 + \sum_{j=1}^{k-1} \binom{k}{j} (2^j + 2^{k-j} - 2)^2.$$

Proof. The result follows by combining Theorem 2.5 and Remark 2.7 with the definition of the first Zagreb Index.

□

4 Other Indices of k-dprime Divisor Function Graph

The second to the last section of this paper, is dedicated in determining the following topological indices of k-dprime divisor function graph for $k = 3, 4,$ and 5 using their graph representation given in Example 2.2

Second Zagreb Index: $M_2(G) = \displaystyle\sum_{uv \in E(G)} deg(u)deg(v)$

Degree-Distance: $DD(G) = \displaystyle\sum_{\{u,v\} \subseteq V(G)} \big[[deg(u) + deg(v)][d(u,v)]\big]$

Balaban: $J(G) = \displaystyle\frac{m}{\mu + 1} \sum_{\{u,v\} \subseteq E(G)} \big(D_u D_v\big)^{-\frac{1}{2}}$

Gutman: $Gut(G) = \displaystyle\sum_{\{u,v\} \subseteq V(G)} deg(u)deg(v)d(u,v)$

Harmonic: $Hm(G) = \displaystyle\sum_{\{uv\} \subseteq E(G)} \frac{2}{deg(u) + deg(v)}$

$$\textbf{Randic: } R(G) = \sum_{\{uv\}\subseteq E(G)} \frac{1}{\sqrt{deg(u)deg(v)}}$$

$$\textbf{First R-index: } R^1(G) = \sum_{v\in V(G)} \big(r(v)\big)^2$$

$$\textbf{Second R-index: } R^2(G) = \sum_{uv\in E(G)} \big(r(u)r(v)\big)$$

$$\textbf{Third R-index: } R^3(G) = \sum_{uv\in E(G)} \big(r(u)+r(v)\big)$$

$$\textbf{Mostar index: } Mo(G) = \sum_{uv\in E(G)} \big|n_u - n_v\big|.$$

Theorem 4.1. *If Γ_3 denote the graph of 3-dprime divisor function graph, then*

$$J(\Gamma_3) = \frac{19}{26}\left[\frac{52+12\sqrt{70}}{35}\right]$$

$$DD(\Gamma_3) = 338$$

$$Gut(\Gamma_3) = 769$$

$$Hm(\Gamma_3) = \frac{589}{154}$$

$$R^1(\Gamma_3) = 2s^2 + 6t^2$$

$$R^2(\Gamma_3) = s^2 + 12st + 15t^2$$

$$R^3(\Gamma_3) = 14s + 42t$$

where $s = 7\cdot 4^6 + 31$ *and* $t = 7^2\cdot 4^5 + 34$

$$R(\Gamma_3) = \left[\frac{23+12\sqrt{7}}{14}\right]$$

$$M_2(\Gamma_3) = 481$$

$$Mo(\Gamma_3) = 36.$$

Before we present the proof, let us first consider some definitions.

Definition 4.2. For any simple connected graph G and a vertex $v \in V(G)$, the expressions

$$S_v = \Big[\sum_{u\in V(G)} deg(u)\Big] - deg(v)$$

and

$$M_v = \frac{\prod_{u\in V(G)} deg(u)}{deg(v)}$$

are the sum and multiplication degree of v, respectively, whereas the R degree of v is defined as $r(v) = S_v + M_v$. Meanwhile, the first R index of G is $R^1(G) = \sum_{v \in V(G)} \big(r(v)\big)^2$. Then the second R index of G is $R^2(G) = \sum_{uv \in E(G)} [r(u)r(v)]$. Finally, the third R index of G is $R^3(G) = \sum_{uv \in E(G)} [r(u) + r(v)]$.

Proof. We will only show the proof for the Randic index, and first, second, and third R indices of 3-dprime divisor function graph. The other result can be proved similarly.

Based on the definition of the Randic index we have

$$
\begin{aligned}
R(\Gamma_3) &= \sum_{\{uv\} \subseteq E(\Gamma_3)} \frac{1}{\sqrt{deg(u)deg(v)}} \\
&= \frac{1}{2}\Bigg[\sum_{\{1v\} \subseteq E(\Gamma_3)} \frac{1}{\sqrt{deg(1)deg(v)}} + \sum_{\{n_1v\} \subseteq E(\Gamma_3)} \frac{1}{\sqrt{deg(n_1)deg(v)}} \\
&+ \sum_{\{n_2v\} \subseteq E(\Gamma_3)} \frac{1}{\sqrt{deg(n_2)deg(v)}} + \sum_{\{n_3v\} \subseteq E(\Gamma_3)} \frac{1}{\sqrt{deg(n_3)deg(v)}} \Bigg] \\
&= \frac{1}{2}\Bigg[\sum_{\{1v\} \subseteq E(\Gamma_3)} \frac{1}{\sqrt{7deg(v)}} + 3\Big[\sum_{\{n_1v\} \subseteq E(\Gamma_3)} \frac{1}{\sqrt{4deg(v)}} \Big] \\
&+ 3\Big[\sum_{\{n_2v\} \subseteq E(\Gamma_3)} \frac{1}{\sqrt{4deg(v)}} \Big] + \sum_{\{n_3v\} \subseteq E(\Gamma_3)} \frac{1}{\sqrt{7deg(v)}} \Bigg] \\
&= \frac{1}{2}\Bigg[\frac{1}{\sqrt{7}}\Big(\frac{1}{\sqrt{7}} + \frac{6}{\sqrt{4}}\Big) + \frac{3}{\sqrt{4}}\Big(\frac{2}{\sqrt{7}} + \frac{2}{\sqrt{4}}\Big) + \frac{3}{\sqrt{4}}\Big(\frac{2}{\sqrt{7}} + \frac{2}{\sqrt{4}}\Big) + \frac{1}{\sqrt{7}}\Big(\frac{1}{\sqrt{7}} + \frac{6}{\sqrt{4}}\Big) \Bigg].
\end{aligned}
$$

Simplifying the above equation gives the desired result.

Next, we prove the result on the R indices of Γ_3. Using the definition presented earlier for the R indices of a graph, the sum and multiplication degree of each vertex in Γ_3 are

$$
\begin{aligned}
S_1 = S_{n_3} = &\ 38 - 7 = 31 \\
S_{n_1} = S_{n_2} = &\ 38 - 4 = 34 \\
M_1 = M_{n_3} = &\ \frac{7^2 \cdot 4^6}{7} = 7 \cdot 4^6 \\
M_{n_1} = M_{n_2} = &\ \frac{7^2 \cdot 4^6}{4} = 7^2 \cdot 4^5,
\end{aligned}
$$

respectively. Letting $s = r(1) = r(n_3) = 7 \cdot 4^6 + 31$ and $t = r(n_1) = r(n_2) = 7^2 \cdot 4^5 + 34$, then, by the definition of the first R index, we get

$$
\begin{aligned}
R^1(\Gamma_3) &= \big(r(1)\big)^2 + 3\Big[\sum_{v \in V(\Gamma_3)} \big(r(n_1)\big)^2 \Big] + 3\Big[\sum_{v \in V(\Gamma_3)} \big(r(n_2)\big)^2 \Big] + \big(r(n_3)\big)^2 \\
R^1(\Gamma_3) &= s^2 + 3(t^2) + 3(t^2) + s^2 \\
R^1(\Gamma_3) &= 2s^2 + 6t^2.
\end{aligned}
$$

Using the formula for the second R index,we obtain the following results:

$$
\begin{aligned}
R^2(\Gamma_3) &= \frac{1}{2}\Bigg[\Big[\sum_{1v \in E(\Gamma_3)} r(1)r(v) \Big] + 3\Big[\sum_{n_1v \in E(\Gamma_3)} r(n_1)r(v) \Big] \\
&\quad + 3\Big[\sum_{n_2v \in E(\Gamma_3)} r(n_2)r(v) \Big] + \Big[\sum_{n_3v \in E(\Gamma_3)} r(n_3)r(v) \Big] \Bigg] \\
R^2(\Gamma_3) &= \frac{1}{2}\big[s(s+6t) + 3t(2s+5t) + 3t(2s+5t) + s(s+6t) \big] \\
R^2(\Gamma_3) &= s^2 + 12st + 15t^2.
\end{aligned}
$$

Lastly, for the third R index, we have

$$R^3(\Gamma_3) = \frac{1}{2}\Big[\Big[\sum_{1v\in E(\Gamma_3)} r(1)+r(v)\Big] + 3\Big[\sum_{n_1v\in E(\Gamma_3)} r(n_1)+r(v)\Big] + 3\Big[\sum_{n_2v\in E(\Gamma_3)} r(n_2)+r(v)\Big] + \Big[\sum_{n_3v\in E(\Gamma_3)} r(n_3)+r(v)\Big]\Big]$$

$$R^3(\Gamma_3) = \frac{1}{2}[[(s+s)+6(s+t)]+3[2(t+s)+5(t+t)]+3[2(t+s)+5(t+t)]+[(s+s)+6(s+t)]]$$

$$R^3(\Gamma_3) = 14s+42t.$$

□

Theorem 4.3. *If Γ_4 denote the graph of 4-dprime divisor function graph, then*

$$J(\Gamma_4) = \frac{65}{102}\left[\frac{202+16\sqrt{330}+66\sqrt{10}+60\sqrt{33}}{165}\right]$$

$$DD(\Gamma_4) = 3712$$

$$Gut(\Gamma_4) = 10557$$

$$Hm(\Gamma_4) = \frac{36367}{4830}$$

$$R^1(\Gamma_4) = 2s^2+8t^2+6w^2$$

$$R^2(\Gamma_4) = s^2+16st+6sw+15t^2+24tw$$

$$R^3(\Gamma_4) = 24s+70t+30w$$

where $s = 15\cdot 8^8\cdot 6^6+115,\ t = 15^2\cdot 8^7\cdot 6^6+122,\ w = 15^2\cdot 8^8\cdot 6^5+124$

$$R(\Gamma_4) = \left[\frac{47+60\sqrt{3}+12\sqrt{10}+8\sqrt{30}}{30}\right]$$

$$M_2(\Gamma_4) = 3993$$

$$Mo(\Gamma_4) = 268.$$

Proof. We will only show the proof for the Gutman index and Harmonic index. The other results can be proved similarly.

From the definition of the Harmonic index, and the properties of Γ_4 we get

$$\begin{aligned} Hm(\Gamma_4) &= \sum_{\{uv\}\subseteq E(\Gamma_4)} \frac{2}{deg(u)+deg(v)} \\ &= \frac{1}{2}\left[\binom{4}{0}\left[\sum_{\{1v\}\subseteq E(\Gamma_4)} \frac{2}{deg(1)+deg(v)}\right] + \binom{4}{1}\left[\sum_{\{n_1v\}\subseteq E(\Gamma_4)} \frac{2}{deg(n_1)+deg(v)}\right]\right. \\ &+ \binom{4}{2}\left[\sum_{\{n_2v\}\subseteq E(\Gamma_4)} \frac{2}{deg(n_2)+deg(v)}\right] + \binom{4}{3}\left[\sum_{\{n_3v\}\subseteq E(\Gamma_4)} \frac{2}{deg(n_3)+deg(v)}\right] \\ &\left.+ \binom{4}{4}\left[\sum_{\{n_4v\}\subseteq E(\Gamma_4)} \frac{2}{deg(n_4)+deg(v)}\right]\right] \end{aligned}$$

$$= \frac{1}{2}\Bigg[\Bigg[\sum_{\{1v\}\subseteq E(\Gamma_4)} \frac{2}{15+deg(v)}\Bigg] + 4\Bigg[\sum_{\{n_1v\}\subseteq E(\Gamma_4)} \frac{2}{8+deg(v)}\Bigg] + 6\Bigg[\sum_{\{n_2v\}\subseteq E(\Gamma_4)} \frac{2}{6+deg(v)}\Bigg]$$

$$+ 4\Bigg[\sum_{\{n_3v\}\subseteq E(\Gamma_4)} \frac{2}{8+deg(v)}\Bigg] + \Bigg[\sum_{\{n_4v\}\subseteq E(\Gamma_4)} \frac{2}{15+deg(v)}\Bigg]\Bigg]$$

$$= \frac{1}{2}\Bigg[\Bigg[\left(\frac{2}{15+15}\right)(1) + \left(\frac{2}{15+8}\right)(8) + \left(\frac{2}{15+6}\right)(6)\Bigg] + \Bigg[\left(\frac{2}{8+15}\right)(2) + \left(\frac{2}{8+8}\right)(3) + \left(\frac{2}{8+6}\right)(3)\Bigg]$$

$$+ \Bigg[\left(\frac{2}{6+15}\right)(2) + \left(\frac{2}{6+8}\right)(4)\Bigg] + \Bigg[\left(\frac{2}{8+15}\right)(2) + \left(\frac{2}{8+8}\right)(3) + \left(\frac{2}{8+6}\right)(3)\Bigg]$$

$$+ \Bigg[\left(\frac{2}{15+15}\right)(1) + \left(\frac{2}{15+8}\right)(8) + \left(\frac{2}{15+6}\right)(6)\Bigg]\Bigg]$$

$$= \frac{36367}{4830}$$

Similarly, for the Gutman index, we obtain the following

$$Gut(\Gamma_4) = \sum_{\{u,v\}\subseteq V(G)} deg(u)deg(v)d(u,v)$$

$$= \frac{1}{2}\Bigg[\binom{4}{0}\Bigg[\sum_{\{1,v\}\subseteq V(\Gamma_4)} deg(1)deg(v)d(1,v)\Bigg] + \binom{4}{1}\Bigg[\sum_{\{n_1,v\}\subseteq V(\Gamma_4)} deg(n_1)deg(v)d(n_1,v)\Bigg]$$

$$+ \binom{4}{2}\Bigg[\sum_{\{n_2,v\}\subseteq V(\Gamma_4)} deg(n_2)deg(v)d(n_2,v)\Bigg] + \binom{4}{3}\Bigg[\sum_{\{n_3,v\}\subseteq V(\Gamma_4)} deg(n_3)deg(v)d(n_3,v)\Bigg]$$

$$+ \binom{4}{4}\Bigg[\sum_{\{n_4,v\}\subseteq V(\Gamma_4)} deg(n_4)deg(v)d(n_4,v)\Bigg]\Bigg]$$

$$= \frac{1}{2}\Bigg[\Bigg[\sum_{\{1,v\}\subseteq V(\Gamma_4)} 15deg(v)(1)\Bigg] + 4\Bigg[\sum_{\{n_1,v\}\subseteq V(\Gamma_4)} 8deg(v)d(n_1,v)\Bigg]$$

$$+ 6\Bigg[\sum_{\{n_2,v\}\subseteq V(\Gamma_4)} 6deg(v)d(n_2,v)\Bigg] + 4\Bigg[\sum_{\{n_3,v\}\subseteq V(\Gamma_4)} 8deg(v)d(n_3,v)\Bigg]$$

$$+ \Bigg[\sum_{\{n_4,v\}\subseteq V(\Gamma_4)} 15deg(v)(1)\Bigg]\Bigg]$$

$$= \frac{1}{2}\Bigg[15\Bigg[\sum_{\{1,v\}\subseteq V(\Gamma_4)} deg(v)\Bigg] + 4(8)\Bigg[\sum_{\{n_1,v\}\subseteq V(\Gamma_4)} deg(v)d(n_1,v)\Bigg] + 6(6)\Bigg[\sum_{\{n_2,v\}\subseteq V(\Gamma_4)} deg(v)d(n_2,v)\Bigg]$$

$$+ 4(8)\Bigg[\sum_{\{n_3,v\}\subseteq V(\Gamma_4)} deg(v)d(n_3,v)\Bigg] + 15\Bigg[\sum_{\{1,v\}\subseteq V(\Gamma_4)} deg(v)\Bigg]\Bigg].$$

Since the distance between any two distinct vertices in Γ_4 is just 1 or 2, then

$$Gut(\Gamma_4) = \frac{1}{2}\Bigg[15\big[1(15)(1) + 8(8)(1) + 6(6)(1)\big] + 32\big[2(15)(1) + 3(8)(1) + 4(8)(2) + 3(6)(1) + 3(6)(2)\big]$$

$$+ 36\big[2(15)(1) + 4(8)(1) + 4(8)(2) + 5(6)(2)\big]\Bigg] + 32\big[2(15)(1) + 3(8)(1) + 4(8)(2) + 3(6)(1) + 3(6)(2)\big]$$

$$+ 15\big[1(15)(1) + 8(8)(1) + 6(6)(1)\big]$$

$$= 3712.$$

□

Theorem 4.4. *If* Γ_5 *denote the graph of* 5*-dprime divisor function graph, then*

$$J(\Gamma_5) = \frac{211}{362}\left[\frac{19353 + 260\sqrt{1426} + 920\sqrt{403} + 1550\sqrt{598}}{9269}\right]$$

$$DD(\Gamma_5) = 19682$$

$$Gut(\Gamma_5) = 124201$$

$$Hm(\Gamma_5) = \frac{45901681}{3106324}$$

$$R^1(\Gamma_5) = 2s^2 + 10t^2 + 20w^2$$

$$R^2(\Gamma_5) = s^2 + 20st + 30sw + 20t^2 + 100tw + 30w^2$$

$$R^3(\Gamma_5) = 52s + 160t + 190w$$

where $s = 31 \cdot 16^{10} \cdot 10^{20} + 391$, $t = 31^2 \cdot 16^9 \cdot 10^{20} + 406$ *and* $w = 31^2 \cdot 16^{10} \cdot 10^{19} + 412$

$$R(\Gamma_5) = \left[\frac{531 + 20\sqrt{31} + 16\sqrt{310} + 310\sqrt{10}}{124}\right]$$

$$M_2(\Gamma_4) = 47401$$

$$Mo(\Gamma_5) = 1720.$$

Proof. We will only show the proof for the Degree-distance Index and second Zagreb Index of Γ_5. The other indices can be proved similarly.

From the definition of the second Zagreb index, we have

$$\begin{aligned}
M_2(\Gamma_5) = \frac{1}{2}\Bigg[&\binom{5}{0}\Bigg[\sum_{1v\in E(\Gamma_5)} deg(1)deg(v)\Bigg] + \binom{5}{1}\Bigg[\sum_{n_1v\in E(\Gamma_5)} deg(n_1)deg(v)\Bigg] \\
&+ \binom{5}{2}\Bigg[\sum_{n_2v\in E(\Gamma_5)} deg(n_2)deg(v)\Bigg] + \binom{5}{3}\Bigg[\sum_{n_3v\in E(\Gamma_5)} deg(n_3)deg(v)\Bigg] \\
&+ \binom{5}{4}\Bigg[\sum_{n_4v\in E(\Gamma_5)} deg(n_4)deg(v)\Bigg] + \binom{5}{5}\Bigg[\sum_{n_5v\in E(\Gamma_5)} deg(n_5)deg(v)\Bigg]\Bigg]\Bigg] \\
&= \frac{1}{2}\Bigg[\Bigg[\sum_{1v\in E(\Gamma_5)} 31deg(v)\Bigg] + 5\Bigg[\sum_{n_1v\in E(\Gamma_5)} 16deg(v)\Bigg] + 10\Bigg[\sum_{n_2v\in E(\Gamma_5)} 10deg(v)\Bigg] \\
&+ 10\Bigg[\sum_{n_3v\in E(\Gamma_5)} 10deg(v)\Bigg] + 5\Bigg[\sum_{n_3v\in E(\Gamma_5)} 16deg(v)\Bigg] + \Bigg[\sum_{n_5v\in E(\Gamma_5)} 31deg(v)\Bigg]\Bigg] \\
&= \frac{1}{2}\Big[\big[31 + 5(16) + 10(10)\big] + 5(16)\big[2(31) + 4(16) + 10(10)\big] + 10(10)\big[2(31) + 5(16) + 3(10)\big] \\
&+ 10(10)\big[2(31) + 5(16) + 3(10)\big] + 5(16)\big[2(31) + 4(16) + 10(10)\big] + \big[31 + 5(16) + 10(10)\big]\Big] \\
&= 47401.
\end{aligned}$$

For the Degree-distance index ofΓ_5, we have

$$
\begin{aligned}
DD(\Gamma_5) &= \frac{1}{2}\Bigg[\binom{5}{0}\Bigg[\sum_{\{1,v\}\subseteq V(\Gamma_5)}[deg(1)+deg(v)]d(1,v)\Bigg] + \binom{5}{1}\Bigg[\sum_{\{n_1,v\}\subseteq V(\Gamma_5)}[deg(n_1)+deg(v)]d(n_1,v)\Bigg] \\
&+ \binom{5}{2}\Bigg[\sum_{\{n_2,v\}\subseteq V(\Gamma_5)}[deg(n_2)+deg(v)]d(n_2,v)\Bigg] + \binom{5}{3}\Bigg[\sum_{\{n_3,v\}\subseteq V(\Gamma_5)}[deg(n_3)+deg(v)]d(n_3,v)\Bigg] \\
&+ \binom{5}{4}\Bigg[\sum_{\{n_4,v\}\subseteq V(\Gamma_5)}[deg(n_4)+deg(v)]d(n_4,v)\Bigg] + \binom{5}{5}\Bigg[\sum_{\{n_5,v\}\subseteq V(\Gamma_5)}[deg(n_5)+deg(v)]d(n_5,v)\Bigg]\Bigg] \\
&= \frac{1}{2}\Bigg[\Bigg[\sum_{\{1,v\}\subseteq V(\Gamma_5)}[31+deg(v)](1)\Bigg] + 5\Bigg[\sum_{\{n_1,v\}\subseteq V(\Gamma_5)}[16+deg(v)]d(n_1,v)\Bigg] \\
&+ 10\Bigg[\sum_{\{n_2,v\}\subseteq V(\Gamma_5)}[10+deg(v)]d(n_2,v)\Bigg] + 10\Bigg[\sum_{\{n_3,v\}\subseteq V(\Gamma_5)}[10+deg(v)]d(n_3,v)\Bigg] \\
&+ 5\Bigg[\sum_{\{n_4,v\}\subseteq V(\Gamma_5)}[16+deg(v)]d(n_4,v)\Bigg] + \Bigg[\sum_{\{n_5,v\}\subseteq V(\Gamma_5)}[31+deg(v)](1)\Bigg]\Bigg] \\
&= \frac{1}{2}\Big[\Big[(31+31)(1)+10(31+16)(1)+20(31+10)(1)\Big] \\
&+ 5\Big[2(16+31)(1)+4(16+16)(1)+5(16+16)(2)+10(16+10)(1)+10(16+10)\Big] \\
&+ 10\Big[2(10+31)(1)+5(10+16)(1)+5(10+16)(2)+3(10+10)(1)+16(10+10)\Big] \\
&+ 10\Big[2(10+31)(1)+5(10+16)(1)+5(10+16)(2)+3(10+10)(1)+16(10+10)\Big] \\
&+ 5\Big[2(16+31)(1)+4(16+16)(1)+5(16+16)(2)+10(16+10)(1)+10(16+10)\Big] \\
&+ \Big[(31+31)(1)+10(31+16)(1)+20(31+10)(1)\Big]\Big] \\
&= 62(31)+460(16)+1040(10) = 19682
\end{aligned}
$$

□

5 Conclusion and Some Problems

In this paper, we introduced the concept of k-dprime divisor function graph and determined some of its basic properties. The general formula for its Wiener, hyper-Wiener, Harary, and First Zagreb index were also presented. We then computed other topological indices of the k-dprime divisor function graph for $k = 3, 4, 5$.

Since this is an introductory paper about k-dprime divisor function graph, there are so many possible problems that the reader might consider. Some possible problems are (1) finding a general closed formula in determining the indices of k-dprime divisor function graph that were presented in Section 4, and (2) studying the energy and distance-eigenvalues of the k-dprime divisor function graph.

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