6th International Conference on
Combinatorics, Cryptography, Computer Science and Computing
November. 17-18, 2021

Mostar Index of Conical and Generalized gear graph

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Abstract

In theoretical chemistry, topological index play a significant role. Bond-additive index have been utilized more extensively than other topological indices that quantify graph peripherality. In this study, we compute the exact formula of one of the recently introduced bond-additive topological index called Mostar Index to the conical graph $C(\ell, k)$. Using the result obtained here, leads to disprove the result obtained by *Colakoğlu* Havare. Moreover we obtained the Mostar index to the new graph called generalized gear graph $C^*(\ell, 2k)$.

Keywords: Conical graph, Gear graph, Mostar index.
Mathematics Subject Classification [2010]: 05C12, 05C05, 05C76

1 Introduction & Preliminaries

Graphs enable us to describe a wide range of systems whose structure and function are determined by the connection pattern of their constituent pieces. A vast range of numerical values, variously called as structural invariants, topological index, or molecular descriptors, have been developed and explored over the last decades in attempt to deduce and condense the data contained in network connection patterns. This phenomena is best noticed in mathematical chemistry, namely chemical graph theory. It sparked a lot of interest, both in the context of various networks and in more traditional applications of chemical graph theory. Some of the most well-known bond-additive distance-based index include the Mostar index, which measures the peripherality of individual bonds (i.e., edges) and then accumulates the contributions of all edges to obtain a global measure of the peripherality of a particular graph. Graph consider in this paper are finite, connected and simple. The vertex and edge set of graph G are denoted $V(G)$ and $E(G)$. The Wiener index was initially defined in terms of edge contributions. For trees T the Wiener index is defined as Some well-known distance based molecular descriptors include,[11, 18] one of the oldest topological indices and most investigated is the Wiener index, which was introduced by Wiener in 1947 to study boiling points of paraffins is defined as

$$W(T) = \sum_{e=uv\epsilon E(T)} n_e^T(u) n_e^T(v),$$

where $n_e^T(u)$ denotes the number of vertices of T closer to u than to v and $n_e^T(v)$ is defined analogously, $G = T$, vice versa. Based on the considerable success of Wiener index, the interesting edge contribution index called Szeged index and PI index has been introduced and studied its various applications see [1, 8, 17, 26]. Recently another bond-additive topological index called the Mostar index was introduced by Došlić et al.[9] is defined as

$$Mo(G) = \sum_{e=uv\epsilon E(G)} \left|n_e^G(u) - n_e^G(v)\right|$$

It attracted considerable attention, in the context of complex networks and in more classical applications of chemical graph theory.(see[2, 4, 7, 10, 14, 15, 16, 19, 20, 21, 25, 23]). Graph operations play an important role in chemical graph theory. Different molecular graphs can be obtained by applying graph operations on some general or particular graphs.F or example, carbon nanostructures, circumscribed donut benzenoid systems and hypercube plane etc.. Hence it is important to study the various graph operations in order to understand how it is related to the corresponding topological index of the original graphs. There are several other results regarding various topological indices under different graph operations are available in the literature[3, 5]. Recently, Kandan et al.[19, 20, 21, 22] obtained PI, Szeged, weighted PI and Weighted Szeged indices for conical and generalized gear graph. Ayache et al.[3] introduced and studied some topological indices of the conical graph (Generalized wheel graph) that consists of center o and (ℓ, k)-cycles $C_k^1, C_k^2, ..., C_k^\ell$ interposed as it is illustrated in Figure 1. and is denoted by $G(\ell, k) = C(\ell, k)$. In this research, we get the exact formula of the Mostar index for the conical graph as well as the newly described generalised gear graph. Using the result obtained here, we obtained the correct value of Mostar index to the wheel graph, which leads to disprove the result obtained in[6].

Definition 1.1. The conical graph $C(\ell, k)$ is a graph which is obtained by taking adjacency from a center vertex o to the first layer of the Cartesian product of C_k and P_ℓ, with $\ell \geq 1$ and $k \geq 3$. (Figure 1.)

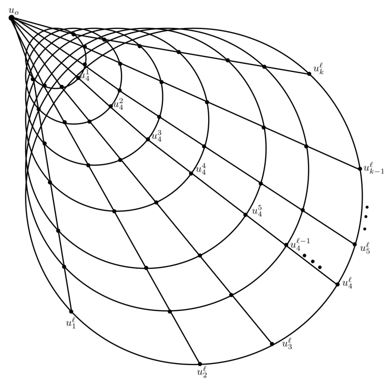

Figure 1: Conical graph

One can be seen that for $\ell = 1$, the graph $C(1, k)$ is not other of classic wheel graph W_k formed by connecting a single vertex o to all the vertices of cycle C_k.
Let the vertex set of $C(\ell, k)$ can be written as $V(C) = \{u_o, u_1^1, ..., u_k^1, ..., u_1^\ell, ..., u_k^\ell\}$ and for our convenience the edge set of $C(\ell, k)$ into four sets such that $E(C) = \bigcup_{n=0}^{3} E_n(C)$, where $E_o(C) = \{u_o u_1^1, u_o u_2^1, ..., u_o u_k^1\}$, $E_1(C) = \{u_1^{\ell-1} u_1^\ell, ..., u_k^{\ell-1} u_k^\ell\}$, $E_2(C) = \{u_1^\ell u_2^\ell, u_2^\ell u_3^\ell, ..., u_k^\ell u_1^\ell\}$, $E_3(C) = E^+ \bigcup E' \bigcup E^*$, where $E^+(C) = \{u_1^1 u_2^1, u_2^1 u_3^1, ..., u_k^1 u_1^1\}$, $E'(C) = \{u_1^j u_2^j, u_2^j u_3^j, ..., u_k^j u_1^j | j = 2, ..., \ell - 1\}$ and $E^*(C) = \{u_1^j u_1^{j+1}, ..., u_k^j u_k^{j+1} | j = 1, ..., \ell - 2\}$ $|E_3(C)| = k + k(\ell - 2) + k(\ell - 2)$. Its clear that for a conical graph $C(\ell, k)$, we have $|V(C(\ell, k))| = k\ell + 1$ and $|E(C(\ell, k))| = 2k\ell$.

2 Main Result

In this section, we obtain the exact value of Mostar index to the conical graph $G(\ell, k)$. For our convenience the conical graph $G(\ell, k)$ to the cycle denoted by $C(\ell, k)$, or simply C.

2.1 Conical graph

The proof of the following lemma follows easily from the above defined edge partitions and structure of the conical graph $C(\ell, k)$. This lemma is used in the proof of the main theorem of this section.

Lemma 2.1. *For a conical graph $C(\ell,k)$, with $\ell \geq 1$ and $k \geq 4$, we have*
(i) if $e = u_o u_i^1 \epsilon E_o(C)$, then $n_e^C(u_o) = \ell(k-3)+1$ and $n_e^C(u_i^1) = \ell$, for $i = 1,2,...,k$
(ii) if $e = u_i^{\ell-1}u_i^{\ell} \epsilon E_1(C)$, then $n_e^C(u_i^{\ell-1}) = (k\ell+1)-k$ and $n_e^C(u_i^{\ell}) = k$, for $i = 1,2,...,k$
(iii) if $e = u_i^{\ell}u_{i+1}^{\ell} \epsilon E_2(C)$, $i = 1(=k+1),2,...,k$ then,
(a) $n_e^C(u_i^{\ell}) = \frac{k\ell}{2} = n_e^C(u_{i+1}^{\ell})$, for k even (b) $n_e^C(u_i^{\ell}) = \frac{(k-1)\ell}{2} = n_e^C(u_{i+1}^{\ell})$, for k odd
(iv) (a) for $i = 1(=k+1),2,...,k$, if $e = u_i^1 u_{i+1}^1 \epsilon E^+(C)$ then $n_e^C(u_i^1) = 2\ell = n_e^C(u_{i+1}^1)$
(b) for $j = 2,...,\ell-1$ and $i = 1(=k+1),2,...,k$, if $e = u_i^j u_{i+1}^j \epsilon E'(C)$
then (i) if $n_e^C(u_i^j) = \frac{k\ell}{2} = n_e^C(u_{i+1}^j)$ k is even
(ii) $n_e^C(u_i^j) = \frac{(k-1)\ell}{2} = n_e^C(u_{i+1}^j)$ k is odd
(c) for $j = 1,2,...,\ell-2$ and $i = 1,2,...,k$, if $e = u_i^j u_i^{j+1} \epsilon E^*(C)$
then $n_e^C(u_i^j) = \sum_{j=1}^{\ell-2}(jk+1)$ and $n_e^C(u_i^{j+1}) = \sum_{j=1}^{\ell-2}(\ell-j)k$.

Using the Lemma 2.1, next we determine the explicit formula of Mostar index to the conical graph $C(\ell,k)$.

Theorem 2.2. *For a conical graph $C(\ell,k)$ with $\ell \geq 1$ and $k \geq 4$, we have*

$$Mo(C(\ell,k)) = \begin{cases} k\left((\ell(k-4)+1)+k((\ell-2)+1)\right)+\frac{(k(\ell-2))^2}{2} & \text{if } \ell \text{ is even} \\ k\left((\ell(k-4)+1)+(k(\ell-2)+1)\right)+k\left((k\ell-1)+\frac{k(\ell-1)(\ell-5)}{2}\right) & \text{if } \ell \text{ is odd.} \end{cases}$$

Proof. By the definition of Mostar index, to obtain it for the conical graph $C(\ell,k)$, we have $Mo(C(\ell,k)) = \sum_{e=uv\epsilon E(C)} \left|n_e^C(u)-n_e^C(v)\right|$. As in the beginning of this section, we partition the edge set of conical graph $C(\ell,k)$ into four sets E_o, E_1, E_2 and E_3, and by the Lemma 2.1, we have
Case(i): For $i = 1,2,3,...,k$, if $e = u_o u_i^1 \epsilon E_o(C)$, then

$$\sum_{e\epsilon E_o(C)} \left|n_e^C(u_o)-n_e^C(u_i^1)\right| = \sum_{e\epsilon E_o(C)} |\ell(k-3)+1-\ell| = k(\ell(k-4)+1), since\ k \geq 4$$

Case(ii): For $i = 1,2,...,k$, if $e = u_i^{\ell-1}u_i^{\ell} \epsilon E_1(C)$, then

$$\sum_{e\epsilon E_1(C)} \left|n_e^C(u_i^{\ell-1})-n_e^C(u_i^{\ell})\right| = \sum_{e\epsilon E_1(C)} |(\ell k+1)-k-k| = k(k(\ell-2)+1), since\ k \geq 4$$

Case(iii): For $i = 1(=k+1),2,...,k$, if $e = u_i^{\ell}u_{i+1}^{\ell} \epsilon E_2(C)$, then

$$\sum_{e\epsilon E_2(C)} \left|n_e^C(u_i^{\ell})-n_e^C(u_{i+1}^{\ell})\right| = \begin{cases} \sum_{e\epsilon E_2(C)} \left|\frac{k\ell}{2}-\frac{k\ell}{2}\right| & \text{if k is even} \\ \sum_{e\epsilon E_2(C)} \left|\frac{(k-1)\ell}{2}-\frac{(k-1)\ell}{2}\right| & \text{if k is odd} \end{cases} = 0$$

Case(iv): For $E_3(C) = E^+(C) \bigcup E'(C) \bigcup E^*(C)$, then the three cases are
sub-case(a): For $i = 1(=k+1),2,...,k$, if $e = u_i^1 u_{i+1}^1 \epsilon E^+(C)$, then

$$\sum_{e\epsilon E^+(C)} \left|n_e^C(u_i^1)-n_e^C(u_{i+1}^1)\right| = \sum_{e\epsilon E^+(C)} |2\ell-2\ell| = 0$$

sub-case(b): For $i = 1(=k+1),2,...,k$ and $j = 2,3,...,\ell-1$, if $e = u_i^j u_{i+1}^j \epsilon E'(C)$, then

$$\sum_{e\epsilon E'(C)} \left|n_e^C(u_i^j)-n_e^C(u_{i+1}^j)\right| = k\begin{cases} \sum_{e\epsilon E_2(C)} \left|\frac{k\ell}{2}-\frac{k\ell}{2}\right| & \text{if k is even} \\ \sum_{e\epsilon E_2(C)} \left|\frac{(k-1)\ell}{2}-\frac{(k-1)\ell}{2}\right| & \text{if k is odd} \end{cases} = 0$$

sub-case(c): For $i = 1, 2, ..., k$ and $j = 1, 2, ..., \ell - 2$, if $e = u_i^j u_i^{j+1} \epsilon E^*(C)$, then

$$\sum_{e \epsilon E^*(C)} \left| n_e^C(u_i^j) - n_e^C(u_i^{j+1}) \right| = \sum_{i=1}^{k} \sum_{j=1}^{\ell-2} |(jk+1) - (\ell - j)k|$$
$$= k \sum_{j=1}^{\ell-2} |k\ell - (2jk+1)|$$

$$= k \begin{cases} \sum\limits_{j=1}^{\frac{\ell-2}{2}} k\ell - (2jk+1) + \sum\limits_{j=\frac{\ell}{2}}^{\ell-2} (2jk+1) - k\ell & \text{if } \ell \text{ is even} \\ \sum\limits_{j=1}^{\frac{\ell-1}{2}} k\ell - (2jk+1) + \sum\limits_{j=\frac{\ell+1}{2}}^{\ell-2} (2jk+1) - k\ell & \text{if } \ell \text{ is odd} \end{cases}$$
$$= k \begin{cases} \frac{k(\ell-2)^2}{2} & \text{if } \ell \text{ is even} \\ k\ell - 1 + \frac{k(\ell-1)(\ell-5)}{2} & \text{if } \ell \text{ is odd} \end{cases}$$

Hence from the above four cases, we have the explicit formula of Mostar index to the conical graph $C(\ell, k)$.

$$Mo(C(\ell,k)) = Mo(E_o(C)) + Mo(E_1(C)) + Mo(E_2(C)) + Mo(E_3(C))$$
$$Mo(C(\ell,k)) = k(\ell(k-4)+1) + k(k(\ell-2)+1) + 0 + 0 + 0 + Mo(E^*)$$
$$Case(i) : For\ \ell\ is\ even$$
$$Mo(C(\ell,k)) = k(\ell(k-4)+1) + k(k(\ell-2)+1) + \frac{(k(\ell-2))^2}{2}$$
$$Case(ii) : For\ \ell\ is\ odd$$
$$Mo(C(\ell,k)) = k\left((\ell(k-4)+1) + (k(\ell-2)+1)\right) + k\left((k\ell-1) + \frac{k(\ell-1)(\ell-5)}{2}\right)$$

□

Using the Theorem 2.2, the following corollaries gives the corrected version to the result proved by $Colako\breve{g}lu\,Havare$ in [6] state that the Mostar index of the wheel W_k is $k(k-4)$ and W_{2k} is $4k(k-2)$.

Corollary 2.3. *For $\ell = 1$ and $k \geq 4$, the wheel graph W_k, whose Mostar index is $Mo(W_k) = k(k-3)$.*

Corollary 2.4. *For $\ell = 1$ and $k \geq 4$, the wheel graph W_k, whose Mostar index is $Mo(W_{2k}) = 2k(2k-3)$.*

For $k = 3$, observe that $Mo(W_3) = 0$.

2.2 Generalized Gear Graph

In this section we obtain the exact value of Mostar Index to the recently introduced graph called generalized gear graph. In[12] Gao and Shi determine the Wiener index and Hyper-Wiener index of gear fan graph and gear wheel graph. More topological indices to the gear related graph and their characterization has been studied see[13, 24]. Motivated by the structure of gear and conical graph, recently Kandan and subramanian [19, 20, 21, 22] introduced generalized gear graph and obtained its PI, Szeged, weighted PI and weighted Szeged indices.

Definition 2.5. The generalized gear graph $C^*(\ell, 2k)$ is obtained by adding one vertex in every two adjacent vertices of the wheel cycle C_k in $C(\ell, k)$ with $\ell \geq 1$ and $k \geq 2$. (see Figure 2.)

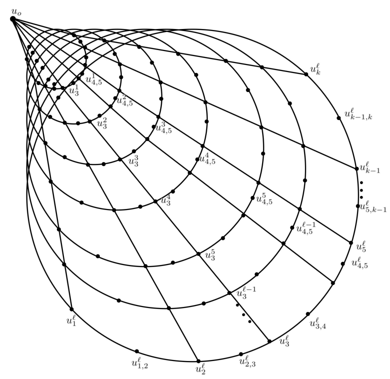

Figure 2: Generalized Gear Graph $C^*(\ell, 2k)$

Let the vertex set of $C^*(\ell, 2k)$ as $V(C^*) = \{u_o, u_1^1, ..., u_k^1, u_1^2, ..., u_k^2, ..., u_1^\ell, u_{1,2}^\ell, ..., u_k^\ell\}$ and for our convenience the edge set of $C^*(\ell, 2k)$ into four set namely $E_o(C^*) = \{u_o u_1^1, u_o u_2^1, ..., u_o u_k^1\}$,
$E_1(C^*) = \Big\{u_1^{\ell-1}u_1^\ell, u_2^{\ell-1}u_2^\ell, u_3^{\ell-1}u_3^\ell, ..., u_k^{\ell-1}u_k^\ell\Big\}$,$E_2(C^*) = \Big\{u_1^\ell u_{1,2}^\ell, u_{1,2}^\ell u_2^\ell, u_2^\ell u_{2,3}^\ell, ..., u_k^\ell u_{k,1}^\ell, u_{k,1}^\ell u_1^\ell\Big\}$,
$E_3(C^*) = E^+(C^*) \bigcup E^-(C^*) \bigcup E^*(C^*)$, where $E^+(C^*) = \Big\{u_1^1 u_{1,2}^1, u_{1,2}^1 u_2^1, ..., u_k^1 u_{k,1}^1, u_{k,1}^1 u_1^1\Big\}$,
$E^-(C^*) = \Big\{u_1^j u_{1,2}^j, ..., u_k^j u_{k,1}^j, u_{k,1}^j u_1^j | j = 2, ..., \ell-1\Big\}$, and $E^*(C^*) = \Big\{u_1^j u_1^{j+1}, ..., u_k^j u_k^{j+1} | j = 1, 2, ..., \ell-2\Big\}$
such that $E(G) = \bigcup\limits_{n=0}^{3} E_n$. Its clear that for a generalized gear graph $C^*(\ell, 2k)$, we have $|V(C^*(\ell, 2k))| = 2k\ell + 1$ and $|E(C^*(\ell, 2k))| = |E_o(C^*)| + |E_1(C^*)| + |E_2(C^*)| + |E_3(C^*)| = k + k + 2k + 3k\ell - 4k = 3k\ell$. Note that if $\ell = 1$, the generalized gear graph $C^*(\ell, 2k)$ is a gear graph, also sometimes known as a bipartite wheel graph.The following Lemma is used to prove the main result of this section which follows immediately from the Figure 2.

Lemma 2.6. *For a generalized gear graph $C^*(\ell, 2k)$, with $\ell \geq 1$ and $k \geq 2$, we have*
(i) For $i = 1, 2, ..., k$, if $e = u_o u_i^1 \epsilon E_o(C^*)$, then $n_e^{C^*}(u_o) = \ell(2k-3) + 1$ and $n_e^{C^*}(u_i^1) = 3\ell$,
(ii) For $i = 1, 2, ..., k$, if $e = u_i^{\ell-1} u_i^\ell \epsilon E_1(C^*)$, then $n_e^{C^*}(u_i^{\ell-1}) = 2k(\ell-1) + 1$ and $n_e^{C^*}(u_i^\ell) = 2k$
(iii) For $i = 1(= k+1), 2, ..., k$ and let $E_2(C^*) = E'(C^*) \bigcup E''(C^*)$, where $e = u_i^\ell u_{i,i+1}^\ell \epsilon E'(C^*)$ and $e = u_{i,i+1}^\ell u_{i+1}^\ell \epsilon E''(C^*)$
sub-case(a) if $e = u_i^\ell u_{i,i+1}^\ell \epsilon E'(C^*)$, then $n_e^{C^*}(u_i^\ell) = \ell(k+1)$ and $n_e^{C^*}(u_{i,i+1}^\ell) = \ell(k-1) + 1$
sub-case(b) if $e = u_{i,i+1}^\ell u_{i+1}^\ell \epsilon E''(C^*)$, then $n_e^{C^*}(u_{i,i+1}^\ell) = \ell(k-1) + 1$ and $n_e^{C^*}(u_{i+1}^\ell) = \ell(k+1)$
(iv) For $e \in E_3(C^*) = E^+(C^*) \bigcup E^-(C^*) \bigcup E^*(C^*)$, we have
Case(a) For $i = 1(= k+1), 2, ..., k$, and let $E^+(C^*) = E_a^+(C^*) \bigcup E_b^+(C^*)$, where $e = u_i^1 u_{i,i+1}^1 \epsilon E_a^+(C^*)$ and $e = u_{i,i+1}^1 u_{i+1}^1 \epsilon E_b^+(C^*)$ then
sub-case(a) if $e = u_i^1 u_{i,i+1}^1 \epsilon E_a^+(C^*)$, then $n_e^{C^*}(u_i^1) = 2\ell(k-1)$ and $n_e^{C^*}(u_{i,i+1}^1) = 2\ell + 1$,
sub-case(b) if $e = u_{i,i+1}^1 u_{i+1}^1 \epsilon E_b^+(C^*)$, then $n_e^{C^*}(u_{i,i+1}^1) = 2\ell + 1$ and $n_e^{C^*}(u_{i+1}^1) = 2\ell(k-1)$
Case(b) For $i = 1(= k+1), 2, ..., k$ and $j = 2, ..., \ell-1$, let $E^-(C^*) = E_a^-(C^*) \bigcup E_b^-(C^*)$, where $e = u_i^j u_{i,i+1}^j \epsilon E_a^-(C^*)$ and $e = u_{i,i+1}^j u_{i+1}^j \epsilon E_b^-(C^*)$
sub-case(a) if $e = u_i^j u_{i,i+1}^j \epsilon E_a^-(C^*)$, then $n_e^{C^*}(u_i^j) = \ell(k+1)$ and $n_e^{C^*}(u_{i,i+1}^j) = \ell(k-1) + 1$,
sub-case(b) if $e = u_{i,i+1}^j u_{i+1}^j \epsilon E_b^-(C^*)$, then $n_e^{C^*}(u_{i,i+1}^j) = \ell(k-1) + 1$ and $n_e^{C^*}(u_{i+1}^j) = \ell(k+1)$,
Case(c) For $i = 1, 2, ..., k$ and $j = 1, 2, ..., \ell-2$, if $e = u_i^j u_i^{j+1} \epsilon E^*(C^*)$
then $n_e^{C^*}(u_i^j) = \sum\limits_{j=1}^{\ell-2}(2kj+1)$ *and* $n_e^{C^*}(u_i^{j+1}) = \sum\limits_{j=1}^{\ell-2} 2k(\ell-j)$.

Using the Lemma 2.6, next we determine the explicit formula of Mostar index to the generalized gear graph $C^*(\ell, 2k)$.

Theorem 2.7. *For a generalized gear graph $C^*(\ell, 2k)$ with $\ell \geq 2$ and $k \geq 3$, we have*

$$Mo(C^*(\ell,2k)) = \begin{cases} k(2\ell(k-3)+1)+2k(2k\ell-(4\ell+1))+k(2k(\ell-2)+1) & \\ +2k(\ell-1)(2\ell-1)+k^2(\ell-2)^2 & \textit{if } \ell \textit{ is even.} \\ k(2\ell(k-3)+1)+2k(2k\ell-(4\ell+1))+k(2k(\ell-2)+1) & \\ +2k(\ell-1)(2\ell-1)+k\left((2k\ell-1)+k(\ell-1)(\ell-5)\right) & \textit{if } \ell \textit{ is odd.} \end{cases}$$

Proof. By the definition of Mostar index, to obtain it for the generalized gear graph $C^*(\ell,2k)$, we have $Mo(C^*(\ell,2k)) = \sum\limits_{e=uv\epsilon E(C^*)} \left|n_e^{C^*}(u) - n_e^{C^*}(v)\right|$. As in the beginning of this section, we partition the edge set of generalized graph $C^*(\ell,2k)$ into four sets E_o, E_1, E_2 and E_3 and by the Lemma 2.6, we have
Case(i): For $i = 1,2,...,k$, if $e = u_o u_i^1 \epsilon E_o(C^*)$, then

$$\sum_{e\epsilon E_o(C^*)} \left|n_e^{C^*}(u_o) - n_e^{C^*}(u_i^1)\right| = \sum_{e\epsilon E_o(C^*)} |(\ell(2k-3)+1)-(3\ell)| = k(2\ell(k-3)+1), since\ k > 2$$

Case(ii): For $i = 1,2,...,k$, if $e = u_i^{\ell-1} u_i^{\ell} \epsilon E_1(C^*)$, then

$$\begin{aligned} &\sum_{e\epsilon E_1(C^*)} \left|n_e^{C^*}(u_i^{\ell-1}) - n_e^{C^*}(u_i^{\ell})\right| \\ =&\sum_{e\epsilon E_1(C^*)} |2k\ell-2k+1-2k| = k(2k(\ell-2)+1), since\ \ell \geq 2, k \geq 2 \end{aligned}$$

Case(iii): For $i = 1(=k+1),2,..,k$, if $e \in E_2(C^*) = E^{'}(C^*) \bigcup E^{''}(C^*)$, with $e = u_i^{\ell} u_{i,i+1}^{\ell} \epsilon E^{'}(C^*)$ and $e = u_{i,i+1}^{\ell} u_{i+1}^{\ell} \epsilon E^{''}(C^*)$, then

$$\begin{aligned} &\sum_{e\epsilon E_2(C^*)} \left|n_e^{C^*}(u) - n_e^{C^*}(v)\right| \\ =&\sum_{e\epsilon E^{'}(C^*)} \left|n_e^{C^*}(u_i^{\ell}) - n_e^{C^*}(u_{i,i+1}^{\ell})\right| + \sum_{e\epsilon E^{''}(C^*)} \left|n_e^{C^*}(u_{i,i+1}^{\ell}) - n_e^{C^*}(u_{i+1}^{\ell})\right| \\ =&k\left|\ell(k+1)-(\ell(k-1)+1)\right| + k\left|\ell(k-1)+1-\ell(k+1)\right| \\ =&2k(2\ell-1) \end{aligned}$$

Case(iv): For $E_3(C^*) = E^+(C^*) \bigcup E^-(C^*) \bigcup E^*(C^*)$ then the three cases are
Case(a): For $i = 1(=k+1),2,...,k$, if $e \in E^+(C^*) = E_a^+(C^*) \bigcup E_b^+(C^*)$, with $e = u_i^1 u_{i,i+1}^1 \epsilon E_a^+(C^*)$ and $e = u_{i,i+1}^1 u_{i+1}^1 \epsilon E_b^+(C^*)$, then

$$\begin{aligned} &\sum_{e\epsilon E^+(C^*)} \left|n_e^{C^*}(u) - n_e^{C^*}(v)\right| \\ =&\sum_{e\epsilon E_a^+(C^*)} \left|n_e^{C^*}(u_i^1) - n_e^{C^*}(u_{i,i+1}^1)\right| + \sum_{e\epsilon E_b^+(C^*)} \left|n_e^{C^*}(u_{i,i+1}^1) + n_e^{C^*}(u_{i+1}^1)\right| \\ =&k\left|(2\ell(k-1)-(2\ell+1)\right| + k\left|(2\ell+1)-2\ell(k-1)\right| \\ =&2k(2k\ell-(4\ell+1)) \end{aligned}$$

Case(b): For $i = 1(=k+1),2,...,k$ and $j = 2,...,\ell-1$, if $e \in E^-(C^*) = E_a^-(C^*) \bigcup E_b^-(C^*)$, with

$e = u_i^j u_{i,i+1}^j \epsilon E_a^-(C^*)$ and $e = u_{i,i+1}^j u_{i+1}^j \epsilon E_b^-(C^*)$, then

$$\begin{aligned}
&\sum_{e\epsilon E^-(C^*)} \left| n_e^{C^*}(u) - n_e^{C^*}(v) \right| \\
&= \sum_{e\epsilon E_a^-(C^*)} \left| n_e^{C^*}(u_i^j) - n_e^{C^*}(u_{i,i+1}^j) \right| + \sum_{e\epsilon E_b^-(C^*)} \left| n_e^{C^*}(u_{i,i+1}^j) - n_e^{C^*}(u_{i+1}^j) \right| \\
&= k(\ell-2)\left|\ell(k+1) - (\ell(k-1)+1)\right| + k(\ell-2)\left|\ell(k-1)+1-\ell(k+1)\right| \\
&= k(\ell-2)\left|(k\ell+\ell) - (k\ell-\ell+1)\right| + k(\ell-2)\left|(k\ell-\ell+1)-(k\ell+\ell)\right| \\
&= 2k(2\ell-1)(\ell-2)
\end{aligned}$$

Case(c): For $i = 1, 2, ..., k$ and $j = 1, 2, ..., \ell - 2$, if $u_i^j u_i^{j+1} \epsilon E^*(C^*)$, then

$$\begin{aligned}
\sum_{e=uv\epsilon E^*(C^*)} \left| (n_e^{C^*}(u_i^j) - n_e^{C^*}(u_i^{j+1}) \right| &= \sum_{i=1}^{k} \left| \sum_{j=1}^{\ell-2} (2kj+1) - \sum_{j=1}^{\ell-2} 2k(\ell-j) \right| \\
&= k \sum_{j=1}^{\ell-2} \left|(2kj+1) - 2k(\ell-j)\right| \\
&= k \sum_{j=1}^{\ell-2} \left|4kj+1-2k\ell\right| \\
&= k \begin{cases} \sum\limits_{j=1}^{\frac{\ell-2}{2}} 2k\ell - (4jk+1) + \sum\limits_{j=\frac{\ell}{2}}^{\ell-2} (4jk+1) - 2k\ell & \text{if } \ell \text{ is even} \\ \sum\limits_{j=1}^{\frac{\ell-1}{2}} 2k\ell - (4jk+1) + \sum\limits_{j=\frac{\ell+1}{2}}^{\ell-2} (4jk+1) - 2k\ell & \text{if } \ell \text{ is odd} \end{cases} \\
&= k \begin{cases} k(\ell-2)^2 & \text{if } \ell \text{ is even} \\ (2k\ell-1) + k(\ell-1)(\ell-5) & \text{if } \ell \text{ is odd} \end{cases}
\end{aligned}$$

Hence from the above four cases, we have the explicit formula of Mostar index to the generalized gear graph $C^*(\ell, 2k)$.

$$\begin{aligned}
Mo(C^*(\ell,2k)) =& Mo(E_o(C^*)) + Mo(E_1(C^*)) + M_o(E_2(C^*)) + Mo(E_3(C^*)) \\
=& k(2\ell(k-3)+1) + k(2k(\ell-2)+1) + 2k(2\ell-1) + 2k(2k\ell-(4\ell+1)) \\
&+ 2k(\ell-2)(2\ell-1) + Mo(E^*(C^*))
\end{aligned}$$

$Case(i): For\ \ell\ is\ even$

$$\begin{aligned}
Mo(C^*(\ell,2k)) =& k(2\ell(k-3)+1) + 2k(2k\ell-(4\ell+1)) + k(2k(\ell-2)+1) + 2k(\ell-1)(2\ell-1) \\
&+ k^2(\ell-2)^2
\end{aligned}$$

$Case(ii): For\ \ell\ is\ odd$

$$\begin{aligned}
Mo(C^*(\ell,2k)) =& k(2\ell(k-3)+1) + 2k(2k\ell-(4\ell+1)) + k(2k(\ell-2)+1) + 2k(\ell-1)(2\ell-1) \\
&+ k\left((2k\ell-1) + k(\ell-1)(\ell-5)\right)
\end{aligned}$$

□

Using the Theorem 2.7, we have following corollary.

Corollary 2.8. *For $\ell = 1$ and $k \geq 3$, the gear graph $C^*(1, 2k)$ whose Mostar index $Mo(C^*(1, 2k)) = 3k(2k - 5)$.*

we obtained the correct value of Mostar index to the wheel graph, which leads to disprove the result obtained by $Colako\breve{g}lu\,Havare$, in[6].

Conclusion

In this paper we obtained the exact value of Mostar index to the conical and generalized gear graph, which leads to disprove the result obtained by $Colako\breve{g}lu\,Havare$. This work may extended for various graph operations and to the molecular structures.

Acknowledgment

The authors thank the anonymous reviewers for their helpful suggestions. P.Kandan is thankful to Tamil Nadu State Council for Higher Education,Chennai for support through research grant under Minor Research Project Scheme 2020-2021: Rc.No.2026/2020 A.

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