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# A combinatorial relation on Fibonacci numbers

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**ABSTRACT**

The sequence of Fibonacci numbers can also be extended to negative index using the re-arranged recurrence relation

$$F_{n-2} = F_n - F_{n-1}.$$

We have $\ldots, F_{-5} = 5, F_{-4} = -3, F_{-3} = 2, F_{-1} = -1, F_0 = 0, F_1 = 1, F_2 = 1, \ldots$

In this article we prove that:

$F_n = (-1)^n F_{-n} = \frac{1}{2}\left[-\binom{n}{1} F_0 + \binom{n}{2} F_1 - \cdots + (-1)^n \binom{n}{n-1} F_{n-1}\right]$, when $n$ is an even number.

And also,

$\left[-\binom{n}{1} F_0 + \binom{n}{2} F_1 - \cdots + (-1)^n \binom{n}{n-1} F_{n-1}\right] = 0$, when $n$ is an odd number.

**KEYWORDS:** Combinatorics, Fibonacci number, Financial analysis, binomial expansion.

## 1. Introduction

Fibonacci ( c. 1170 – c. 1240–50), was an Italian mathematician from the Republic of Pisa, considered to be "the most talented Western mathematician of the Middle Ages". Fibonacci popularized the Hindu–Arabic numeral system in the Western world primarily through his composition in 1202 of Liber Abaci (Book of Calculation). He also introduced Europe to the sequence of Fibonacci numbers, which he used as an example in Liber Abaci.

In mathematics, the Fibonacci numbers, denoted by $F_n$ , form a sequence, called the Fibonacci sequence, such that each number is the sum of the two preceding ones, starting from 0 and 1. That is, $F_{0,} = 0$, $F_{1,} = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n > 1$.

The beginning of the sequence is thus

$$\{0,1,1,2,3,5,8,13,21,34,55,89,144,\ldots\}.$$

Liber Abaci posed and solved a problem involving the growth of a population of rabbits based on idealized assumptions. The solution, generation by generation, was a sequence of numbers later known as Fibonacci numbers. Although Fibonacci's Liber Abaci contains the earliest known description of the sequence outside of India, the sequence had been described by Indian mathematicians as early as the sixth century.

There is a strong relation between Fibonacci numbers and golden ratio, where

$$\frac{1+\sqrt{5}}{2} = \varphi \simeq 1.618\ldots \quad , \frac{1-\sqrt{5}}{2} \simeq -0.618\ldots$$

Johannes Kepler proved that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi$ .

A Fibonacci strategy for day trading forex uses a series of numbers, ratios and patterns to establish entry and exit points. We'll explain how to use Fibonacci retracement levels and extensions to identify support and resistance areas, plus profit taking targets. In finance, Fibonacci retracement is a method of technical analysis for determining support and resistance levels. Fibonacci retracement are a popular technical analysis tool that help traders to identify future price movement. The basic premise is that in a market uptrend, you buy on a retracement at a Fibonacci support level, while during a downtrend, you sell at a Fibonacci resistance level. They are named after their use of the Fibonacci sequence. Fibonacci retracement is based on the idea that markets will retrace a predictable portion of a move, after which they will continue to move in the original direction.

According to Fibonacci, any nature-driven market, such as the financial market, is prone to make retracements that are either 0.618 (61.8%) or 0.382 (38.2%) of the distance a stock, currency, or index has moved. There are four Fibonacci tools have been able to utilize the golden ratio. The next step is supplementing your forex trading strategy with extension levels. Extensions use Fibonacci numbers and patterns to determine profit taking points. Extensions continue past the 100% mark and indicate possible exits in line with the trend. For the purposes of using Fibonacci numbers for day trading forex, the key extension points consist of 61.8%, 261.8% and 423.6%.

Numerous other identities can be derived using various methods. Here are some of them:

Cassini's identity states that:

$$(F_n)^2 - F_{n+1}F_{n-1} = (-1)^{n-1}.$$

Catalan's identity is a generalization:

$$(\mathrm{F_n})^2 - \mathrm{F_{n+r}F_{n-r}} = (-1)^{\mathrm{n}-1}(\mathrm{F_r})^2.$$

d'Ocagne's identity

$$F_{n+1}F_m - F_{m+1}F_n = (-1)^n F_{m-\mathrm{n}}.$$

The last is an identity for doubling $n$; other identities of this type are:

$$5(F_n)^3 + 3(-1)^n F_n = F_{3n}.$$

These can be found experimentally using lattice reduction, and are useful in setting up the special number field sieve to factorize a Fibonacci number. More generally:

$$F_{kn+\mathrm{c}} = \sum_{i=0}^{k} \binom{k}{i} F_{c-\mathrm{i}} (F_n)^{\mathrm{i}} (F_{n+1})^{\mathrm{k-i}}$$

or alternatively,

$$F_{kn+\mathrm{c}} = \sum_{i=0}^{k} \binom{k}{i} F_{c+\mathrm{i}} (F_n)^{\mathrm{i}} (F_{n-1})^{\mathrm{k-i}}.$$

In this article we try to use some combinatorial relations and find new formulas based on the combinatorial relations of Fibonacci numbers. We will discuss this in more detail in the next section.

For more result, see [1], [2], [3], [4] , [5] and [6].

## 2 MAIN RESULTS

In the previous section we gave a detailed introduction to Fibonacci numbers and their applications. In the last part of the introduction, we also mentioned some of the recursive relations that exist between Fibonacci numbers.. In this section, we try to find new generalizations of these relations.

Notice that the sequence of Fibonacci numbers can also be extended to negative index using the re-arranged recurrence relation

$$F_{n-2} = F_n - F_{n-1}.$$

We have $\dots, \mathrm{F}_{-5} = 5, \mathrm{F}_{-4} = -3, \mathrm{F}_{-3} = 2, \mathrm{F}_{-1} = -1, \mathrm{F}_0 = 0, \mathrm{F}_1 = 1, \mathrm{F}_2 = 1, \dots$

We obtain with a simple calculation:

$$F_3 = 1 - \binom{2}{1} F_0 + \binom{2}{2} F_1.$$

$$F_4 = 1 - \binom{3}{1} F_0 + \binom{3}{2} F_1 - \binom{3}{3} F_2.$$

$$F_5 = 1 - \binom{4}{1} F_0 + \binom{4}{2} F_1 - \cdots \binom{4}{3} F_2 + \binom{4}{4} F_3.$$

After mentioning this preliminary example, we prepare to prove the main theorem.

**Main Theorem**. Assume that $F_n$, for $n \in \mathbb{Z}$, be the general sentence of the Fibonacci sequence.

Then:

$F_n = (-1)^n F_{-n} = \frac{1}{2}\left[-\binom{n}{1} F_0 + \binom{n}{2} F_1 - \cdots + (-1)^n \binom{n}{n-1} F_{n-1}\right]$, when $n$ is an even number.

And also,

$\left[-\binom{n}{1} F_0 + \binom{n}{2} F_1 - \cdots + (-1)^n \binom{n}{n-1} F_{n-1}\right] = 0$, when $n$ is an odd number.

**Proof.**

First, we claim that

$$F_n = (-1)^n F_{-n} = 1 - \binom{n-1}{1} F_0 + \binom{n-1}{2} F_1 - \cdots + (-1)^{n-1} \binom{n-1}{n-1} F_{n-2}.$$

We know that

$$\frac{1+\sqrt{5}}{2} = \varphi \simeq 1.618 \ldots \quad , \frac{1-\sqrt{5}}{2} \simeq -0.618 \ldots$$

$$F_n = \frac{1}{\sqrt{5}} (\varphi^n - (-\varphi)^{-n}).$$

And $\varphi^{-1} + \mathbf{1} = \varphi$. On the other hand,

$$\textstyle\sum_{k=0}^{n} (-1)^k \binom{n}{k} \varphi^{k-1} = \varphi^{-1}(1-\varphi)^n.$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (-\varphi)^{-k+1} = -\varphi(1+\varphi^{-1})^n$$

Hence,

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k \varphi^{k-1} - (-1)^k \binom{n}{k} (-\varphi)^{-k+1} = \varphi^{-1}(1-\varphi)^n - \varphi(1+\varphi^{-1})^n.$$

$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\varphi^{k-1}-(-1)^k\binom{n}{k}(-\varphi)^{-k+1}=-(-\varphi)^{-n-1}+(\varphi)^{n+1}=\sqrt{5}\,F_{n+1}.$

Therefore,

$$F_n=(-1)^nF_{-n}=1-\binom{n-1}{1}F_0+\binom{n-1}{2}F_1-\cdots+(-1)^{n-1}\binom{n-1}{n-1}F_{n-2}.$$

As we claim.

Now, notice that

$$\binom{n+1}{k}=\binom{n}{k}+\binom{n-1}{k-1}.$$

On the other hand, we have $\mathrm{F}_{\mathrm{n}-2}+\mathrm{F}_{\mathrm{n}-1}=\mathrm{F}_{\mathrm{n}}$. Now, by using the placement of the above relation, we conclude that

$F_n=(-1)^nF_{-n}=\frac{1}{2}\left[-\binom{n}{1}F_0+\binom{n}{2}F_1-\cdots+(-1)^n\binom{n}{n-1}F_{n-1}\right]$, when $n$ is an even number.

And also,

$\left[-\binom{n}{1}F_0+\binom{n}{2}F_1-\cdots+(-1)^n\binom{n}{n-1}F_{n-1}\right]=0$, when $n$ is an odd number.

As we claim.

□

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