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Numerical solution of a class of third-kind Volterra integral equations using Ritz approximation

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Abstract

The manuscript deals with a numerical method of a class of third-kind Volterra integral equations by the ritz method. To use this method, we transform the third-kind Volterra integral equations into an optimization problem and obtain the system of nonlinear algebraic equations. By polynomial basis functions, we approximate solutions. Then we have the coefficients of polynomial expansions by solving the system of nonlinear equations. Some numerical examples are included to demonstrate the theoretical results and the performance of the numerical approximation.

Keywords: Integral equations; Third-kind Volterra integral equations; Ritz approximation; Polynomial basis functions
Mathematics Subject Classification [2010]: 13D45, 39B42

1 Introduction

In this manuscript, we consider the following Volterra integral equation (VIE).

$$t^{\beta}u(t) = f(t) + \int_0^t (t-x)^{-\alpha}\kappa(t,x)u(x)dx, \quad t \in [0,T], \tag{1}$$

where $\beta > 0, \alpha \in [0,1), \alpha+\beta \geq 1, f(t) = t^{\beta}g(t)$ with a continuous function g and κ is continuous on the domain $\Delta := \{(t,x) : 0 \leq x \leq t \leq T\}$ and is of the form

$$\kappa(t,x) = x^{\alpha+\beta-1}\kappa_1(t,x),$$

where κ_1 is continuous on Δ. The existence of the term t^{β} in the left–hand side of (1) gives it special properties, which are distinct from those of VIEs of the second kind (where the left-hand side is always different from zero), and also distinct from those of the first kind (where the left-hand side is constant and equal to zero). This is why in the literature they are often mentioned as VIEs of the third kind. This class of equations has attracted the attention of researchers in the last years. The existence, uniqueness, and regularity of solutions to (1) were discussed in [2]. In that paper, the authors have derived necessary conditions to convert the equation into a cordial VIE, a class of VIEs which was studied in detail in [7, 8]. This made possible to apply to (1) some results known for cordial equations. In particular, the case $\alpha+\beta > 1$ is of special interest, because in this case, if $\kappa_1(t,x) > 0$, the integral operator associated to (1) is not compact and it is not possible to assure the solvability of the equation by classical numerical methods. In [2], the authors have introduced a modified graded mesh and proved that with such mesh the collocation method is applicable and has the same convergence order as for regular equations. In [5], two of the authors of the present paper have applied a different approach, which consisted in expanding the solution as a series of

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adjusted hat functions and approximating the integral operator by an operational matrix. The advantage of that approach is that it reduces the problem to a system of linear equations with a simple structure, which splits into subsystems of three equations, making the method efficient and easy to implement [5]. A limitation of this technique is that the optimal convergence order ($O(h^4)$) can be attained only under the condition that the solution satisfies $u \in C^4([0,T])$, which is not the case in many applications. It is worth remarking here that there is a close connection between equations of class (1) and fractional differential equations [6]. Actually, the kernel of (1) has the same form as the one of a fractional differential equation, and if we consider the case $\kappa_1(t,x) \cong 1$, then the integral operator is the Riemann–Liouville operator of order $1-\alpha$. Therefore, it makes sense to apply to this class of equations numerical approaches that have recently been applied with success to fractional differential equations and related problems [6].

2 Solution of a class of third-kind Volterra integral equations

In this work we focus on a class of third-kind Volterra integral equations: Consider the following VIEs:

$$t^{\beta}u(t) = f(t) + \int_0^t (t-x)^{-\alpha}\kappa(t,x)u(x)dx, \quad t \in [0,T], \tag{2}$$

where the initial condition is as follows:

$$u(0) = u_0, \tag{3}$$

The method consists of conversion third-kind Volterra integral equations to optimization problem and expanding the solution by polynomial basis functions with unknown coefficients.

We approximate $u(t)$ as

$$u(t) \cong \tilde{u}(t) = \sum_{i=0}^{m} c_i t\phi_i(t) + w(t), \tag{4}$$

where $\phi_i(t)$ are polynomial basis functions and c_i are unknown coefficients. In following, we determine $w(t)$ as $w(0) = x_0$.

Now we have the following optimal problem

$$J[c_0, c_1, ..., c_m] = \int\limits_0^1 (t^{\beta}\tilde{u}(t) - f(t) - \int_0^t (t-x)^{-\alpha}\kappa(t,x)\tilde{u}(x)dx)^2 dt, \tag{5}$$

If $c_k$are decided by the optimizing function J, then by (7), we obtain functions which approximate the optimum value of J. To find unknowns c_k, $k = 0, 1, ..., m$ in $\widetilde{u}(t)$, according to the necessary conditions of optimization for (5), we have

$$\frac{\partial J}{\partial c_k} = 0, \quad k = 0, ..., m. \tag{6}$$

Then by solving the above system of m algebraic equations (6), we obtain c_k, $k = 0, 1, ..., m$. The approached demonstrated here relies on the Ritz method. Then with solving this problem by mathematica software, we obtain c_k. The method presented here is based on the Ritz method. We refer the interested reader to [3, 4] for more information.

3 Illustrative examples

To demonstrate the effectiveness of the method, here we consider a of third-kind Volterra integral equations. The following example demonstrate that the desired approximate solution can be determined by solving the resulting system of equations, which can be effectively computed using symbolic computing codes on any personal computer. Illustrative example show that this method in comparison to other methods has high accuracy and is easily implemented.

Example 3.1. As the first example, we consider the following third-kind VIE, which is an equation of Abel type [2, 5]:

$$t^{2/3}u(t) = f(t) + \int_0^t \frac{\sqrt{3}\sqrt[3]{x}u(x)}{\pi(t-x)^{2/3}}\,dx, \tag{7}$$

where

$$f(t) = t^{47/12}\left(1 - \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{55}{12}\right)}{\sqrt{3}\pi\Gamma\left(\frac{59}{12}\right)}\right).$$

The exact solution is

$$u(t) = t^{\frac{13}{4}}.$$

We used the Ritz approximation that in section 2 explained. In first, we determine

$$\tilde{u}(t) = \sum_{i=0}^{m} c_i t^{i+1}. \tag{8}$$

By solved system (6) with different value of m, we obtain c_i coefficients. To replace this coefficients in (7), we obtain approximated solution. Fig. 1 shows the absolute error of this problem obtained by the present method with $m = 12$. From Fig. 1, we can see that the present method provides accurate results.

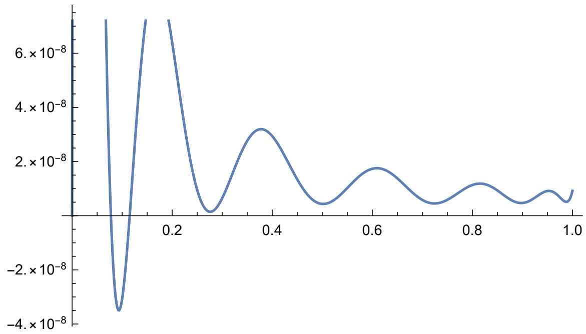

Fig.1. The absolute error between exact and numerical solution for $m = 12$.

The following table shows the values of minimum J for different values of approximations.

	$m = 5$	$m = 7$	$m = 12$
J	1.39473×10^{-12}	1.5103×10^{-14}	-1.06238×10^{-16}

Example 3.2. Consider the following third-kind VIE, which is used in the modelling of some heat conduction problems with mixed-type boundary conditions [2, 5]:

$$tu(t) = \frac{6}{7}t^3\sqrt{t} + \int_0^t \frac{1}{2}u(x)\,dx, \tag{9}$$

This equation has the exact solution $u(t) = t^{\frac{5}{2}}$.
We used the Ritz approximation that in section 2 explained. In first, we determine

$$\tilde{u}(t) = \sum_{i=0}^{m} c_i t^{i+1}. \tag{10}$$

By solved system (6) with different value of m, we obtain c_i coefficients. To replace this coefficients in (7), we obtain approximated solution. Fig. 2 shows the absolute error of this problem obtained by the present method with $m = 12$. From Fig. 2, we can see that the present method provides accurate results.

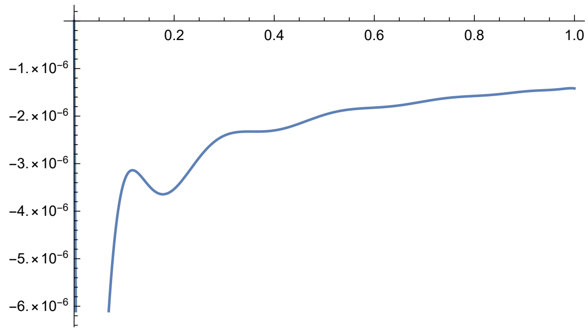

Fig.2. The absolute error between exact and numerical solution for $m = 12$.

The following table shows the values of minimum J for different values of approximations.

	$m=5$	$m=7$	$m=12$
J	9.84621×10^{-12}	2.11886×10^{-13}	1.42109×10^{-14}

Example 3.3. Consider the following VIE of the third kind[2]:

$$t^{3/2}u(t) = f(t) + \int_0^t \frac{\sqrt{2}xu(x)}{2\pi(t-x)^{1/2}}\,dx, \qquad t \in [0,1], \tag{11}$$

where

$$f(t) = t^{33/10}\left(1 - \frac{\Gamma\left(\frac{19}{5}\right)}{\sqrt{2\pi}}\right).$$

This equation has the exact solution $u(t) = t^{\frac{9}{5}}$. We used the Ritz approximation that in section 2 explained. In first, we determine

$$\tilde{u}(t) = \sum_{i=0}^{m} c_i t^{i+1}. \tag{12}$$

By solved system (6) with different value of m, we obtain c_i coefficients. To replace this coefficients in (7), we obtain approximated solution. Fig. 3 shows the absolute error of this problem obtained by the present method with $m = 12$. From Fig. 3, we can see that the present method provides accurate results.

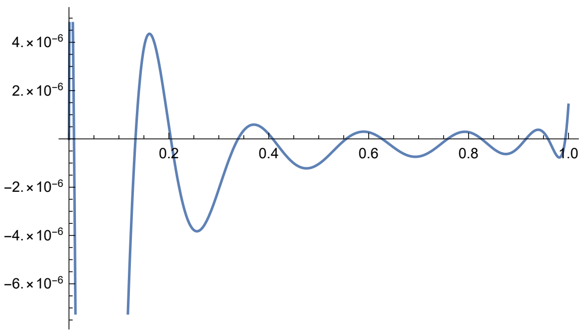

Fig.3. The absolute error between exact and numerical solution for $m = 12$.

The following table shows the values of minimum J for different values of approximations.

	$m=5$	$m=7$	$m=12$
J	7.75946×10^{-12}	2.58085×10^{-13}	5.32215×10^{-14}

4 Conclusion

This paper presents a simple and effective approach to solve a a class of third-kind Volterra integral equations. The desired approximate solution can be determined by solving the resulting system of equations, which can be effectively computed using symbolic computing codes on any personal computer. Illustrative example show that this method has high accuracy and is easily implemented. The method will be expected to deal with other problems such as fractional inverse problems, fractional optimal problems, which will be discussed in a future papers.

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