6th International Conference on
Combinatorics, Cryptography, Computer Science and Computing
November. 17-18, 2021

Power graph of a finite group

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Abstract

The power graph $P(G)$ of a group G is the graph whose vertex set is the group elements and two elements are adjacent if one is a power of the other. In this paper, we consider some graph theoretical properties of a power graph $P(G)$ that can be related to its group theoretical properties.

Keywords: Power graph, clique number, EPPO-group, maximal cyclic subgroup.
Mathematics Subject Classification [2010]: 05C12, 91A43, 05C69.

1 Introduction

All groups and graphs in this paper are finite. Throughout the paper, we follow the terminology and notation of [8, 9] for groups and [11] for graphs.

Groups are the main mathematical tools for studying symmetries of an object and symmetries are usually related to graph automorphisms, when a graph is related to our object. Groups linked with graphs have been arguably the most famous and productive area of algebraic graph theory, see [1, 8] for details. The power graphs is a new representation of groups by graphs. These graphs were first used by Chakrabarty et al. [4] by using semigroups. It must be mentioned that the authors of [4] were motivated by some papers of Kelarev and Quinn [5, 6, 7] regarding digraphs constructed from semigroups. We also encourage interested readers to consult papers by Cameron and his co-workers on power graphs constructed from finite groups [2, 3].

Suppose G is a finite group. The *power graph* $P(G)$ is a graph in which $V(P(G)) = G$ and two distinct elements x and y are adjacent if and only if one of them is a power of the other. If G is a finite group then it can be easily seen that the power graph $P(G)$ is a connected graph of diameter 2. In [4], it is proved that for a finite group G, $P(G)$ is complete if and only if G is a cyclic group of order 1 or p^m, for some prime number p and positive integer m.

Following [9, 10], two finite groups G and H are said to be conformal if and only if they have the same number of elements of each order. In [10], the following question was investigated:

Question: *For which natural numbers n are any two conformal groups of order n isomorphic?*

Let G be a group and $x \in G$. We denote by $o(x)$ the order of x and G is said to be EPO$-$group, if all non-trivial element orders of G are prime. An EPPO$-$group is that its element orders are prime power. The set of all elements order of G is called its *spectrum*, denoted by $\pi_e(G)$, A maximal subgroup H of G is denoted by $H < \cdot\, G$ and the set of all elements of G of order k is denoted by $\Omega_k(G)$.

Suppose Γ is a graph. A subset X of the vertices of Γ is called a *clique* if the induced subgraph on X is a complete graph. The maximum size of a clique in Γ is called the *clique number* of Γ and denoted by $\omega(\Gamma)$. A subset Y of $V(\Gamma)$ is an *independent set* if the induced subgraph on X has no edges. The maximum size

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of an independent set is called the *independence number* of G and denoted by $\alpha(G)$. The *chromatic number* of Γ is the smallest number of colors needed to color the vertices of Γ so that no two adjacent vertices share the same color. This number is denoted by $\chi(\Gamma)$.

Throughout this paper our notation is standard and they are taken from the standard books on graph theory and group theory such as [9, 11].

2 Main results

Suppose G is a finite group of order n. Chakrabarty, Ghosh and Sen [4] proved that the number of edges of $P(G)$ can be computed by the following formula:

$$e = \frac{1}{2}\sum_{a\in G}\{2o(a) - \phi(o(a)) - 1\},$$

where ϕ is the Euler's totient function. In the case that G is cyclic, we have:

$$e = \frac{1}{2}\sum_{d|n}\{2d - \phi(d) - 1\}\phi(d).$$

Moreover, $P(Z_n)$ is nonplanar when $\phi(n) > 7$ or $n = 2^m$, $m \geq 3$. Finally, if $n \geq 3$ then $P(Z_n)$ is Hamiltonian.

Suppose $D(n)$ denotes the set of all positive divisors of n. It is well-known that $(D(n), |)$ is a distributive lattice. $D(n)$ is a Boolean algebra if and only if n is square-free. In the following theorem we apply the structure of this lattice to compute the clique and chromatic number of $P(Z_n)$.

Theorem 2.1. *Suppose G is a group and $A \subseteq G$. The vertices of A constitute a complete subgraph in $P(G)$ if and only if $\{\langle x\rangle \mid x \in A\}$ is a chain.*

Theorem 2.2. *Suppose $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$, where $p_1 < p_2 < ... < p_r$ are prime numbers. Then*

$$\omega(P(Z_n)) = \chi(P(Z_n)) = p_r^{\alpha_r} + \sum_{j=0}^{r-2}(p_{r-j-1}^{\alpha_{r-j-1}} - 1)\left(\prod_{i=0}^{j}\phi(p_{r-i}^{\alpha_{r-i}})\right).$$

Theorem 2.3. *Suppose that G is a full exponent group and $n = Exp(G) = p_1^{\beta_1}p_2^{\beta_2}\cdots p_r^{\beta_r}$, where $p_1 < p_2 < ... < p_r$ are prime numbers. If x is an element of order n then*

$$\omega(P(G)) = \chi(P(G)) = p_r^{\beta_r} + \sum_{j=0}^{r-2}(p_{r-j-1}^{\beta_{r-j-1}} - 1)\left(\prod_{i=0}^{j}\phi(p_{r-i}^{\beta_{r-i}})\right).$$

Corollary 2.4. *Let G be a finite group. Then the power graph $P(G)$ is planar if and only if $\pi_e(G) \subseteq \{1, 2, 3, 4\}$.*

In [4, Lemma 4.7], the authors proved that if G is a cyclic group of order n, $n \geq 3$ and $\phi(n) > n$ then $P(G)$ is not planar. Also, in [4, Lemma 4.8] it is proved that a cyclic group of order 2^n, $n \geq 3$, is not planar. In the following corollary we apply Corollary 4 to find a simple classification for planarity of the power graph of cyclic groups.

Corollary 2.5. *The power graph of a cyclic group of order n is planar if and only if $n = 2, 3, 4$.*

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