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On Norm Properties of Particular Symmetric Matrix Connected Jacobsthal Sequence

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ABSTRACT

In this note we consider a particular symmetric matrix involving Jacobsthal sequence. We investigate determinant, Euclidean norm, 1-Norm and Infinity-Norm of this matrix and give lower bound and some upper bounds for the spectral norm of it. We prove that this matrix is positive definite. Also, we represent some numerical examples about these results.

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KEYWORDS: Symmetric matrix, Euclidean Norm, Spectral norm, Determinant.

1 INTRODUCTION

In modelling some natural phenomena or some mathematical problems that follow regular rules, we encounter the recursive sequences. Fibonacci, for example, became interested in a strange issue in 1202. He wanted to know what the pattern would be if he had a pair of male and female rabbits and defined a behaviour for their descendants. And so, the famous Fibonacci sequence was defined. After that, many authors have studied Fibonacci sequence, some generalization of this sequence and several recurrence sequences of natural numbers like as Lucas, Pell, PellLucas, Pell, Jacobsthal, and Jacobsthal-Lucas sequences and etc.

A Jacobsthal sequence $\{J_n\}$ is defined by

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0,\ J_1 = 1,\ n \geq 0. \quad (1)$$

The first values of Jacobsthal sequence are:

0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731.

The Jacobsthal-Lucas sequence$\{j_n\}$ is defined by

$$j_n = j_{n-1} + 2j_{n-2}, \quad j_0 = 2,\ j_1 = 1,\ n \geq 0. \quad (2)$$

The first values of Jacobsthal-Lucas sequence are:

2, 1, 5, 7, 17, 31, 65, 127, 257, 511, 1025, 2047, 4097, 8191.

In [1] Bueno studied (k,h)-Jacobsthal sequence of the form

$$T_n = kT_{n-1} + 2\mathrm{h}T_{n-2}. \quad (3)$$

He found a formula of nth term and sum of the first n terms of this sequence. In [2] Campus and Catrino established an explicit formula for the term of order n, the well-known Binets formula, Catalans and Ocagnes identities and a generating function for k-Jacobsthal-Lucas sequence. Godase and Dhakne in [7] by using particular 2×2 matrices have studied some properties of k−Fibonacci and k−Lucas numbers. Petoudi and Pirouz in [11] investigated some properties of (k,h)-Pell sequence and (k,h)-Pell-Lucas sequence. For more information about Jacobsthal sequence, Jacobsthal-Lucas sequence, Fibonacci sequence, some generalizations of Fibonacci sequence and other related number sequences we refer to [3]-[6], [12]-[15].

In this paper we consider a particular matrix of the form $\mathbb{J} = [J_{k_{i,j}}]_{i,j=1}^{n}$ where $k_{i,j} = \min(i,j) + 1$ and J_n is the nth Jacobsthal number. In exact, this matrix is given as

$$\mathbb{J} = \begin{bmatrix} J_2 & J_2 & J_2 & \cdots & J_2 \\ J_2 & J_3 & J_3 & \cdots & J_3 \\ J_2 & J_3 & J_4 & \cdots & J_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_2 & J_3 & J_4 & \cdots & J_{n+1} \end{bmatrix}, \qquad (4)$$

Its Hadmard exponential matrix $e^{\circ\mathbb{J}}$ is given by

$$e^{\circ\mathbb{J}} = \begin{bmatrix} e^{J_2} & e^{J_2} & e^{J_2} & \cdots & e^{J_2} \\ e^{J_2} & e^{J_3} & e^{J_3} & \cdots & e^{J_3} \\ e^{J_2} & e^{J_3} & e^{J_4} & \cdots & e^{J_4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{J_2} & e^{J_3} & e^{J_4} & \cdots & e^{J_{n+1}} \end{bmatrix}. \qquad (5)$$

We represent the Euclidean norm of $\mathbb{J}$ and find two upper bounds and lower bounds for the spectral norm of this matrix by using some well-known properties of Jacobsthal sequence.
Also, we prove that $\mathbb{J}$ is a positive definite matrix. We verify that the 1-norm, infinity norm and trace of $\mathbb{J}$ are all equal. Finally, some examples about these results are given in this paper. Some properties of Jacobsthal sequence and Jacobsthal-Lucas sequences are listed here:

Binet formula:

$$J_n = \frac{2^n - (-1)^n}{3}, \qquad j_n = 2^n + (-1)^n. \qquad (6)$$

Summation formulas:

$$\sum_{k=2}^{n} J_k = \frac{J_{n+2} - 3}{2}, \qquad \sum_{k=2}^{n} j_k = \frac{j_{n+2} - 5}{2}. \qquad (7)$$

Let $A = [a_{ij}]$ is an $n \times n$ matrix. Trace of A is denoted by $Trace(A)$ and is defined by

$$Trace(A) = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}. \qquad (8)$$

All definition and statements of this section are in the reference [1]-[4] and [17].

Let $A = [a_{ij}]$ is an $n \times n$ matrix. The $l_p$norm of A is defined by

$$\|A\|_p = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^p\right)^{\frac{1}{p}}. \qquad (9)$$

For $p = 2$, this norm is called Frobenius or Euclidean norm and showed by $\|A\|_E$. The maximum column length norm of A denoted $c(B)$, is defined as

$$c_1(A) = max_j \sqrt{\sum_i |a_{ij}|^2}\ , \qquad (10)$$

and the maximum row length norm of matrix A denoted $r(A)$, is defined

$$r_1(A) = max_i \sqrt{\sum_j |a_{ij}|^2}\,. \qquad (11)$$

Let $B = [b_{ij}]$ and $C = [c_{ij}]$are $m \times n$ matrices. Then Hadamard product of B and C is defined by $B \circ C = [b_{ij} c_{ij}]$. If $A = B \circ C$, (Hadamard product of B and C). Then we have

$$\|A\|_2 \leq r_1(B) c_1(C)\,, \qquad (12)$$

Where $c_1(C)$is the maximum column length norm of C and $r_1(B)$ is the maximum row length norm of B. The spectral norm of A is defined by

$$\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} \rho_i} \qquad (13)$$

where $\rho_i$is the eigenvalue of matrix AA^H and A^H is conjugate transpose of matrix A. There is a relation between euclidean and spectral norm, that is

$$\frac{1}{\sqrt{n}} \|A\|_E \leq \|A\|_2 \leq \|A\|_E. \qquad (14)$$

It is known that

$$\sum_{k=0}^{n-1} x^k = \frac{x^n - 1}{x - 1}\,, \qquad (15)$$

$$\sum_{k=0}^{n-1} k x^k = \frac{(n-1)x^n - n x^{n-1} + 1}{(x-1)^2} \qquad (16)$$

If $A = [a_{ij}]_{i,j=1}^{n}$ is an $n \times n$ matrix, then we have the following relations about the 1-norm and infinity norm A:

$$\|A\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}| \qquad (17)$$

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}| \qquad (18)$$

2 MAIN RESULTS

In this section we give determinants, Euclidean norm, 1-Norm, Infinity-Norm, lower bound and two upper bounds for the spectral norm of matrix $\mathbb{J}$. Also, we show that this matrix is positive definite.

Theorem (2.1). Let $\mathbb{J}$ be a matrix as in (4), then we get

$$\det(\mathbb{J}) = 2 \prod_{k=2}^{n-1} J_{k-1}\,.$$

Proof. Using elementary row operations on (4) we get

$$\det(\mathbb{J}) = \begin{bmatrix} J_2 & J_2 & J_2 & \cdots & J_2 \\ 0 & J_3 - J_2 & J_3 - J_2 & \cdots & J_3 - J_2 \\ 0 & 0 & J_4 - J_3 & \cdots & J_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & J_{n+1} - J_n \end{bmatrix}$$

$$= J_2 \prod_{k=2}^{n-1} (J_{k+1} - J_k) = \prod_{k=2}^{n-1} 2J_{k-1} = 2\prod_{k=2}^{n-1} J_{k-1}.$$

Lemma (2.2). Let J_n be the nth Jacobsthal number. Then

$$\sum_{k=1}^{n} {J_k}^n = \frac{1}{9}[J_{2n+2} + (-1)^{n+1}J_{n+2} + n].$$

Proof. See [1].

Theorem (2.3). Let $\mathbb{J}$ be a matrix as in (4). The upper bound for $\mathbb{J}$ is given as

$$\|\mathbb{J}\|_2 \leq \frac{1}{9}\sqrt{\left([J_{2n+4} + (-1)^{n+2}J_{n+3} + n + 1] - \frac{1}{9}\right)(J_{2n+2} + (-1)^{n+1}J_{n+2} + n)}.$$

Proof. Using definition of Hadamard product for symmetric matrix $\mathbb{J}$, we write

$$\mathbb{J} = \begin{bmatrix} J_2 & 1 & 1 & \cdots & 1 \\ J_2 & J_3 & 1 & \cdots & 1 \\ J_2 & J_3 & J_4 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J_2 & J_3 & J_4 & \cdots & J_{n+1} \end{bmatrix} \circ \begin{bmatrix} 1 & J_2 & J_2 & \cdots & J_2 \\ 1 & 1 & J_3 & \cdots & J_3 \\ 1 & 1 & 1 & \cdots & J_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} = A \circ B.$$

According to (10) and (11) and lemma (2.) we get

$$r_1(A) = max_i \sqrt{\sum\nolimits_j |a_{ij}|^2} = max_i \sqrt{\sum\nolimits_j |a_{ij}|^2} = \sqrt{\sum_{i=2}^{n+1} {J_i}^2} = \sqrt{\sum_{i=1}^{n+1} {J_i}^2 - 1}$$

$$= \sqrt{\frac{1}{9}[J_{2n+4} + (-1)^{n+2}J_{n+3} + n + 1] - 1}\,.$$

Also, we have

$$c_1(B) = max_j \sqrt{\sum\nolimits_i |a_{ij}|^2} = \sqrt{\sum_{i=2}^{n} {J_i}^2 + 1} = \sqrt{\sum_{i=1}^{n} {J_i}^2 - 1 + 1}$$

$$= \sqrt{\sum_{i=1}^{n} {J_i}^2} = \sqrt{\frac{1}{9}[J_{2n+2} + (-1)^{n+1}J_{n+2} + n]} = \frac{1}{3}\sqrt{J_{2n+2} + (-1)^{n+1}J_{n+2} + n}\,.$$

Hence by (12) and after some computation we get

$$\|\mathbb{J}\|_2 \leq r_1(A)c_1(B) = \tfrac{1}{9}\sqrt{\left([J_{2n+4} + (-1)^{n}J_{n+3} + n + 1] - \tfrac{1}{9}\right)(J_{2n+2} + (-1)^{n+1}J_{n+2} + n)}.$$

This proves the theorem.

Remark (2.4). Applying (16) we can write

$$\sum_{k=1}^{n} kx^{k+1} = \frac{\mathrm{n}x^{n+2} - (n+1)x^{n+1} + \mathrm{x}}{(x-1)^2}.$$

Theorem (2.5). Let $\mathbb{J}$ be a matrix as in (4). Then the Euclidean norm of $\mathbb{J}$ is given by

$$\|\mathbb{J}\|_E^2 = \left[\frac{(2n+1)(J_{2n+4}+(-1)^n J_{n+3}+n)}{9}\right] - \left[\frac{16(-2)^n(3\,n+1)+2^{2n+5}(3n-1)+9n(9n+1)+16}{81}\right].$$

Proof. By definition of Euclidean norm, we can write

$$\|\mathbb{J}\|_E^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\left|J_{k_{i,j}}\right|^2 = \sum_{k=1}^{n}(2n-2k+1)\,J_{k+1}^2 = (2n+1)\sum_{k=1}^{n}J_{k+1}^2 - 2\sum_{k=1}^{n}k\,J_{k+1}^2\,. \quad (2.5.1)$$

We have

$$\sum_{k=1}^{n}J_{k+1}^2 = \sum_{k=1}^{n+1}J_k^2 - J_1 = \frac{1}{9}[J_{2n+4}+(-1)^n J_{n+3}+n+1-1]$$
$$= \frac{1}{9}[J_{2n+4}+(-1)^n J_{n+3}+n]. \qquad (2.5.2)$$

By (6) we have $J_n = \frac{2^n-(-1)^n}{3}$. Hence, we obtain

$$\sum_{k=1}^{n}k\,J_{k+1}^2 = \sum_{k=1}^{n}k\left(\frac{2^{k+1}-(-1)^{k+1}}{3}\right)^2 = \frac{1}{9}\left(\sum_{k=1}^{n}k\,[2^{k+1}-(-1)^{k+1}]^2\right)$$

$$= \frac{1}{9}\left(\sum_{k=1}^{n}k\,[4^{k+1}-2(-2)^{k+1}+1]\right).$$

Using remark (2.4) we get

$$\sum_{k=1}^{n}k4^{k+1} = \frac{16[(3n)4^n-4^n+1]}{9}, \sum_{k=1}^{n}k(-2)^{k+1} = -\frac{4}{9}[3(-2)^n n-(-2)^n-1]\,,$$

Also, we have $\sum_{k=1}^{n}k = \frac{n(n+1)}{2}$.

By adding these summations, and after some calculations we obtain

$$\sum_{k=1}^{n}k\,J_{k+1}^2 = \frac{48(-2)^n n+16(-2)^n+(96n)2^{2n}-2^{2n+5}+9n^2+9n+16}{162}. \qquad (2.5.3)$$

Consequently, by (2.5.1), (2.5.2) and (2.5.3) we find that

$$\|\mathbb{J}\|_E^2 = \left[\frac{(2n+1)(J_{2n+4}+(-1)^n J_{n+3}+n)}{9}\right] - \left[\frac{16(-2)^n(3\,n+1)+2^{2n+5}(3n-1)+9n(9n+1)+16}{81}\right].$$

Theorem (2.6). Let $\mathbb{J}$ be a matrix as in (4). Then we have the following lower bound and upper bound for the spectral norm of $\mathbb{J}$.

$$\frac{1}{\sqrt{n}}\sqrt{\frac{(2n+1)(J_{2n+4}+(-1)^n J_{n+3}+n)}{9} - \frac{16(-2)^n(3\,n+1)+2^{2n+5}(3n-1)+9n(9n+1)+16}{81}}$$
$$\le \|\mathbb{J}\|_2$$

$$\leq \sqrt{\frac{(2n+1)(J_{2n+4}+(-1)^n J_{n+3}+n)}{9}-\frac{16(-2)^n(3\,n+1)+2^{2n+5}(3n-1)+9n(9n+1)+16}{81}}.$$

Proof. It can be proved from (14) and theorem (2.5).

Lemma (2.7). Let A is a symmetric matrix, then

(I) A is positive definite if and only if all its leading principal minors are positive.

(II) A is positive definite if and only if all of its eigenvalues are positive.

Proof. See [17].

Theorem (2.8). Let $\mathbb{J}$ be a matrix as in (4). Then $\mathbb{J}$ is a positive definite matrix.

Proof. According to theorem (2.1), all of leading principal minors of $\mathbb{J}$ are positive. Since $\mathbb{J}$ is a symmetric matrix, thus the result follows from lemma (2.7).

Corollary (2.9). Let $\mathbb{J}$ be a matrix as in (4). Then, all the eigenvalues of $\mathbb{J}$ are positive.

Proof. it follows from lemma (2.7) and theorem (2.8).

Theorem (2.10). Let $\mathbb{J}$ be a matrix as in (4). Then 1-Norm, Infinity Norm and Trace of $\mathbb{J}$ are all equal and we have

$$\|\mathbb{J}\|_1 = \|\mathbb{J}\|_\infty = Trace(\mathbb{J}) = \sum_{k=2}^{n} J_k = \frac{J_{n+2}-3}{2}.$$

Proof. According to the definition of 1-norm and the entries of matrix form $\mathbb{J} = [J_{k_{i,j}}]_{i,j=1}^{n}$, we have:

$$\|\mathbb{J}\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{n} J_{k_{i,j}} = J_2 + J_3 + \cdots + J_{n+1} = \sum_{k=2}^{n} J_k .$$

By (7) we know that $\sum_{k=2}^{n} J_k = \frac{J_{n+2}-3}{2}$, thus we deduce that

$$\|\mathbb{J}\|_1 = \frac{J_{n+2}-3}{2}.$$

Using similar manners, we can prove that

$$\|\mathbb{J}\|_\infty = trace(\mathbb{J}) = \sum_{k=2}^{n} J_k = \frac{J_{n+2}-3}{2}.$$

3 NUMERICAL EXAMPLES

In this section we provide two examples about the determinant, Euclidean norm, 1-Norm, Infinity Norm, Trace and bounds for the spectral norm of two particular symmetric matrices involving Jacobsthal sequence of the form (4). We have used the mathematical online compiler in order to obtain some computations about these examples.

Example 1. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 \\ 1 & 3 & 5 & 5 \\ 1 & 3 & 5 & 11 \end{bmatrix}$, then we have the following numerical results about this matrix:

(a1) $Det(A) = 24$,

(a2) $\|A\|_{\mathrm{E}} = 15.7481$,

(a3) Eigenvalues of A =0.5332, 0.9305, 3.1422, 15.3940.

(a4) The 1-Norm, Infinity Norm and trace of matrix A are given as follows:

$$\|A\|_1 = \|A\|_\infty = Trac(A) = 20.$$

(a5) We have the following lower bound and upper bound for the spectral norm of symmetric matrix A:

$$\frac{1}{2}(15.7481) \leq \|A\|_2 \leq 15.7481.$$

Example 2. Consider the matrix $B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 3 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\ 1 & 3 & 5 & 11 & 11 & 11 & 11 & 11 & 11 \\ 1 & 3 & 5 & 11 & 21 & 21 & 21 & 21 & 21 \\ 1 & 3 & 5 & 11 & 21 & 43 & 43 & 43 & 43 \\ 1 & 3 & 5 & 11 & 21 & 43 & 85 & 85 & 85 \\ 1 & 3 & 5 & 11 & 21 & 43 & 85 & 171 & 171 \\ 1 & 3 & 5 & 11 & 21 & 43 & 85 & 171 & 341 \end{bmatrix}$, then we have the following numerical results about this matrix:

(b1) $Det(B) = 3242131200,$

(b2) $\|B\|_{\mathrm{E}} = 508.7721.$

(b3) Eigenvalues of B are 0.5332,0.9290, 2.5894, 5.1896, 10.6530, 21.2752, 43.1246, 100.4145, 496.2915.

(b4) The 1-Norm, Infinity Norm and trace of matrix B are given as follows:

$$\|A\|_1 = \|A\|_\infty = Trac(A) = 681.$$

(b5) We have the following lower bound and upper bound for the spectral norm of symmetric matrix B:

$$\frac{1}{3}(508.7721) \leq \|B\|_2 \leq 508.7721.$$

4 CONCUSION

In this paper we considered a particular symmetric matrix involving Jacobsthal sequence. We proved some relation about the determinant, Euclidean norm, 1-Norm and Infinity-Norm of this matrix. We computed lower bound and some upper bounds for the spectral norm of this matrix. We proved that this matrix is positive definite. Also, we represented two numerical examples about these results.

For the future works, one can consider circulant matrices involving Jacobsthal sequence or other special number sequence relate to Jacobsthal sequence and obtain new results about the norm properties of these matrices.

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REFERENCES

[1] A. C. F. Bueno, A note on (k,h)-Jacobsthal sequences, Math. sci. Lett, 1, 2013, pp.81-87.

[2] H. Campos, P. Catrino, On Some Identities of k-Jacobsthal-Lucas Numbers, Int. Journal of Math. Analysis, Vol. 8, no. 10, 2014, pp. 489 - 494.

[3] A. Coskun, N. Taskara, On the some properties of circulant matrix with third order linear recurrent sequence, 2014, 1-9.

[4] E. Dupree, B.Mathes, Singular values of k-Fibonacci and k-Lucas Hankel matrix, Int. J. Contemp. Math. Science. Vol 7 , 2012, pp. 2327–2339.

[5] A. D. Godase, M. B. Dhakne, On the properties of generalized Fibonacci like polynomials, Int. J. Adv. Appl. Math. and Mech, 2(3), 2015, pp. 234 - 251.

[6] A. D. Godase, M. B. Dhakne, On the properties of k-Fibonacci and k-Lucas numbers, Int. J. Adv. Appl. Math. and Mech, 2(1), 2014, pp. 100 - 106.

[7] A. Coskun, N. Taskara, On the some properties of circulant matrix with third order linear recurrent sequence, 2018, pp.1-9.
[8] K. Kaygisiz, A. Sahin, k-sequences of Generalized Van der Laan and Generalized Perrin Polynomials, Elsevier, https://www.researchgate.net/publication/51956495, 2011, pp.1-20.
[9] A. S. Liana, I. Wloch, Jacobsthal and Jacobsthal Lucas Hybrid numbers, Annales Mathematiccae Silesianae, 33, 2019, pp.276-283.
[10] A. Nalli, M. Sen, On the norms of circulant matrices with generalized Fibonacci numbers, Seluk J. Appl. Math, Vol.11, No.1 2010, pp.107-116.
[11] S. H. J. Petroudi, B. Pirouz, On some properties of (k,h)-Pell sequence and (k,h)-Pell-Lucass sequence, Int. J. Adv. Appl. Math. and Mech, 2015, pp.98–101.
[12] S.H.J. Petroudi, M. Pirouz, On special circulant matrices with (k,h)-Jacobsthal sequence and (k,h)-Jacobsthal-like sequence, Int. J. Mathematics and scientific computation, 2016, pp. 44-47.
[13] S.H.J. Petroudi, M. Pirouz, On circulant matrix involving the Van Der Laan Hybrid sequence, Erzincan university journal of science and technology, 2021, pp.370-381.
[14] S.H.J. Petroudi, M. Pirouz, A. Ozkoc, On Some Properties of Particular Tetranacci sequence, J. Int. Math. Virtual Inst., Vol. 10(2), 2020, pp.361-376.
[15] A. G. Shanon,A. F. Horadam ,P. G. Anderson, Properties of Cordonnier, Perrin and Van Der Laan numbers, International Journal of Mathematical Education in Science and Technology, 2006, pp.825-831.S.
[16] F. Yilmaz, D. Bozkurt, Hessenberg matrices and the Pell and Perrin numbers. J. Number Theory, 2011, pp.1390–1396.
[17] F. Zhang, Matrix theory basic results and techniques, Springer, 2011.