# Derangement problem

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**ABSTRACT**

Assume that $\sigma \epsilon S_n$ is a permutation on $n$ elements, for example $\{1,2,3,\dots,n\}$. Consider that $k$ is an integer such that $1 \le k < n$. We define

$$X_k = \{\sigma \epsilon S_n \mid \sigma(i+1) + k \neq \sigma(i) \; for \; 1 \le i \le n-1\}.$$

Also, assume that $s_k = |X_k|$, the cardinal number of $X_k$.

Then,

$$s_k = \binom{k}{1} s_{k-1} + \cdots + \binom{k}{k-1} s_1 + \binom{k}{k} s_0.$$

When, $D_n = s_0$.

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**KEYWORDS:** Derangements, FPF property, Permutations, Combinatorial method.

## 1 INTRODUCTION

Derangements are arrangements of some number of objects into positions such that no object goes to its specified position. The derangement problem was formulated by P. R. de Montmort in 1708, and solved by him in 1713 (de Montmort 1713-1714). Nicholas Bernoulli also solved the problem using the inclusion-exclusion principle. The number of derangements of an n-element set is called the n -th derangement number or rencontres number, or the sub-factorial of n and is sometimes denoted $D_n$. Counting the derangements of a set amounts to what is known as the hat-check problem, in which one considers the number of ways in which n hats can be returned to n people such that no hat makes it back to its owner. This number satisfies the recurrences

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

Also, it is well-known that

$$\lim_{n\to\infty} \frac{D_n}{n!} = e^{-1} = 0.3678 \ldots$$

The problème des rencontres asks how many permutations of a size $n$ set have exactly $k$ fixed points.

Derangements are an example of the wider field of constrained permutations. For example, the ménage problem asks if n opposite sex couples are seated man-woman-man-woman-... around a table, how many ways can they be seated so that nobody is seated next to his or her partner?

More formally, given sets A and S, and some sets U and V of surjections A $\rightarrow$ S, we often wish to know the number of pairs of functions (f, g) such that f is in U and g is in V, and for all a in $A$, $f(a) \neq g(a)$; in other words, where for each $f$ and $g$, there exists a derangement $\varphi$ of $S$ such that $f(a) = \varphi(g(a))$.

Another generalization is the following problem:

The interesting this is that the number e itself also has applications in probability theory, in a way that is not obviously related to exponential growth. Suppose that a gambler plays a slot machine that pays out with a probability of one in n and plays it n times. Then, for large n, the probability that the gambler will lose every bet is approximately $1/e$.

Recently, Gordon and McMahon in [6] considered isometries of the $n$-dimensional hypercube that leave no facet unmoved. Algebraically, such an isometry is an element $\sigma$ of the hyperoctahedral group $B_n$ for which $\sigma(i) \neq i$ for any $i$. Combinatorially, the problem then is to enumerate $n \times n$ matrices with entries from $\{0, \pm 1\}$ such that each row and column has exactly one nonzero entry and no diagonal entry equals 1. Gordon and McMahon derive a formula for the number of facet derangements, an expression of facet derangements in terms of permutation derangements, and recurrence relations for facet derangements.

Gordon and McMahon noted that the number of derangements in the hyperoctahedral group gives the rising 2-binomial transform of the derangement numbers for $S_n$. More generally, they shows that the cyclic derangement numbers give a mixed version of the rising $r$-binomial transform and falling $(r - 1)$ binomial transform of $D_n$. This new hybrid $k$-binomial transform may share many of the nice properties of Spivey and Steil's transforms, including Hankel invariance and/or a simple description of the change in the exponential generating function. Further, it could be interesting to evaluate the expression for negative or even non-integer values of $k$. For instance, taking $k = 1/2$ gives the binomial mean transform which is of some interest.

Also, in [2], authors defined a special case of derangement as following:

**Definition A.** The FPF property obviously means that in this permutation any cycle is of length greater than one. What we add to this requirement is the following. We take a permutation on $n + r$ letters and we restrict the first r of these to be in distinct cycles. We arrive at the definition of the subject of the paper An FPF permutation on $n + r$ letters will be called FPF r-permutation if in its cycle decomposition the first r letters appear to be in distinct cycles. The number of FPF r-permutations denote by $D_r(n)$ and call r-derangement number. The first r elements, as well as the cycles they are contained in, will be called distinguished. This definition was motivated by the extensive study of the so-called r-Stirling numbers of the first kind which count permutations with a fixed number of cycles where the same restriction on the first distinguished elements is added.

They proved that:

**Theorem B.** For all $n > 2$ and $r > 0$, we have

$$D_r(n) = rD_{r-1}(n-1) + (n-1)D_r(n-2) + (n+r-1)D_r(n-1)$$

In following section, we define a new case of derangement and obtain some result about this special case of derangement.

For more result, see [1], [2], [3], [4] and [5].

## 2 MAIN RESULT

In this section we define a new special case of derangement and also we obtain some relation on this subset of derangements. This special case of derangement is a subset of block derangement.

Assume that $\sigma \epsilon S_n$ is a permutation on n elements, for example $\{1,2,3,\dots,n\}$. Consider that $k$ is an integer such that $0 \leq k < n$. We define

$$X_k = \{\sigma \epsilon S_n \mid \sigma(i+1) + k \neq \sigma(i) \; for \; 1 \leq i \leq n-1\}.$$

Also, assume that $s_k = |X_k|$, the cardinal number of $X_k$. Our main goal in this paper is to find a way to calculate the value of $s_k$. The following theorem will give an inductive method for calculating the number $s_k$.

**Main Theorem**. Assume that $\sigma \epsilon S_n$ is a permutation on $n$ elements, for example $\{1,2,3,\dots,n\}$. Consider that $k$ is an integer such that $1 \leq k < n$. We define

$$X_k = \{\sigma \epsilon S_n \mid \sigma(i+1) + k \neq \sigma(i) \; for \; 1 \leq i \leq n-1\}.$$

Also, assume that $s_k = |X_k|$, the cardinal number of $X_k$.

Then,

$$s_k = \binom{k}{1} s_{k-1} + \dots + \binom{k}{k-1} s_1 + \binom{k}{k} s_0.$$

When, $D_n = s_0$.

**Proof.**

Let's take a closer look at the definition of derangement once again. In fact in calculating the number of derangements, we count the number of permutations that have no fixed points. Now, we will pay more attention to how $X_k$ is defined.

Notice that, when $\sigma(i+1)+k > n$, there is no point in choosing. Therefore, there is no such permutation. In fact, k points have freedom of choice. Here you are facing a special kind of derangements. In fact, it can be shown by creating a one-to-one correspondence that we are dealing with derangements with $n-k$ members. This topic needs more careful consideration that we will deal with it in the next paragraph.

To complete the theorem process, we use induction on k. It is easily seen when $k=0$, induction is established. Notice that $\frac{D_{n+1}}{n} = D_n + D_{n-1}$, and hence, $s_1 = \frac{D_{n+1}}{n}$. Using induction, first suppose that the theorem holds for $i \leq k-1$. According to the induction process, We must prove that the theorem holds for k as well.

By creating a one-to-one correspondence, one can see that in a particular case, in fact, we are going to count the number of members of the following set:

$$T_k = \{\sigma \epsilon S_n \mid \sigma(i) \neq i \; for \; 1 \leq i \leq n-k\}.$$

So, it is enough for us to focus all our efforts on calculating the number of members $T_k$. Note the structure of the set $T_k$. The set $T_k$ is divided into two parts. First part of it is practically a derangement. And the other part is a normal permutation. Now suppose in the second part only the number i members are not transferred to themselves by permutation. Using induction assumption, the number of possible situations in this situation is equal to

$$\binom{k}{i} s_{k-i}.$$

Therefore,

$$s_k = \binom{k}{1} s_{k-1} + \cdots + \binom{k}{k-1} s_1 + \binom{k}{k} s_0.$$

When, $D_n = s_0$, as we claim.

□

**Corollary.** $s_1 = \frac{D_{n+1}}{n}$ and $s_2 = 2\frac{D_{n+1}}{n} + D_n$ .

**Proof.**

Notice that $\frac{D_{n+1}}{n} = D_n + D_{n-1}$. Now, by using the previous theorem, the conclusion is obtained.

□

## 3 ACKNOWLEDGEMENTS

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