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# Colored distance of graphs and some of its applications

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**Abstract**

For a graph facility location problem, each vertex can be considered to be the location of part of a department. One seeks to optimally locate the departments in order to minimize some function of the distances between departments. Consider a factory with a rectangular planar area. The area is divided into a square mesh, a grid with 1-by-l unit pieces, and each piece is assigned to a department. Members of the same department are not required to exchange information, but they must exchange information with every other member of a different department. We seek to minimize the total distance between members of different departments. This problem is a particular instance of the colored distance problem, in which a general graph is colored with $t$ colors and the colored distance is the sum of the distances between vertices of different colors. The dual concept of color distance named partition distance. Generally this problem is NP-hard but there are some algorithms for special graphs that they are polynomial. In this talk, It is showed the optimal colored distance solution for the n-by-m grid graph when all colors are of equal size. Also it is presented an polynomial time algorithm for trees. Finally, we will present some open problem conjectures for future works.

**Keywords:** Colored distance, Partition distance, Average distance, Facility location network

**Mathematics Subject Classification [2010]:** 05C12, 05C05

## 1 Introduction

Let $G$ be a graph and let $\mathcal{P} = \{V_1, \ldots, V_k\}$ be a partition of $V(G)$. Let $f_{\mathcal{P}} : V(G) \to [k] = \{1, \ldots, k\}$ be the index function of $\mathcal{P}$ defined with $f_{\mathcal{P}}(v) = i$, where $v \in V_i$. The *Wiener index* $W(G)$ of $G$, defined as the sum of the distances between all unordered pairs of vertices of $G$, can be decomposed with respect to $\mathcal{P}$ as

$$W(G) = \sum_{\substack{\{u,v\} \\ f_{\mathcal{P}}(u) = f_{\mathcal{P}}(v)}} d_G(u,v) + \sum_{\substack{\{u,v\} \\ f_{\mathcal{P}}(u) \neq f_{\mathcal{P}}(v)}} d_G(u,v)\,.$$

An equivalent approach to the Wiener index of a graph is to study its average distance, Denoting the above sums with $W_{\mathcal{P}}(G)$ and $W_{\overline{\mathcal{P}}}(G)$, respectively, the Wiener index of $G$ thus decomposes as

$$W(G) = W_{\mathcal{P}}(G) + W_{\overline{\mathcal{P}}}(G)\,. \tag{1}$$

We call $W_{\mathcal{P}}(G)$ the *partition distance* of $G$ (with respect to $\mathcal{P}$) [5]. The function $W_{\overline{\mathcal{P}}}(G)$ was earlier introduced by Dankelmann, Goddard, and Slater [1] as the *colored distance* of $G$ (with respect to $\mathcal{P}$) with a location problem from as a motivation. In this problem one aims to partition the nodes of a network considered into a set of facility nodes and a set of customer nodes, such that the average distance between a facility and a customer is minimized. Clearly, if $|\mathcal{P}| = |V(G)|$, then $W_{\overline{\mathcal{P}}}(G) = W(G)$ and if $|\mathcal{P}| = 1$, then $W_{\overline{\mathcal{P}}}(G) = 0$.

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Moreover, the so-called (un-weighted) *median problem* asks to determine a partition $\mathcal{P} = \{V_1, V_2\}$, where $|V_1| = 1$, such that $W_{\overline{\mathcal{P}}}(G)$ is smallest possible.

Several invariants of wide interest in chemical graph theory can be expressed as instances of the partition distance. First of all, if $|\mathcal{P}| = 1$, then $W_{\mathcal{P}}(G) = W(G)$. Consider next a fixed positive integer $k$ and a graph $G$ with vertices $v_1, \ldots, v_n$, where $\deg(v_1) = \cdots = \deg(v_r) = k$ and $\deg(v_i) \neq k$ for any $i > r$. Then setting $\mathcal{P} = \{\{v_1, \ldots, v_r\}, \{v_{r+1}\}, \ldots, \{v_n\}\}$ we have

$$W_{\mathcal{P}}(G) = TW_k(G)\,,$$

where $TW_k(G)$ is the *generalized terminal Wiener index* of $G$. Moreover, some basic graph invariants can also be expressed using the partition distance. Let us give two examples here. If $G$ is a graph with vertices $v_1, \ldots, v_n$, where $d(v_1, v_2) = \operatorname{diam}(G)$, then by setting $\mathcal{P} = \{\{v_1, v_2\}, \{v_3\}, \ldots, \{v_n\}\}$ we have $W_{\mathcal{P}}(G) = \operatorname{diam}(G)$. And if $v_1, \ldots, v_n$ are vertices of $G$ such that $v_1, \ldots, v_{\omega(G)}$ induce a largest clique of $G$, then by setting $\mathcal{P} = \{\{v_1, \ldots, v_{\omega(G)}\}, \{v_{\omega(G)+1}\}, \ldots, \{v_n\}\}$, the clique number $\omega(G)$ of $G$ can be written as $\omega(G) = \left(1 + \sqrt{1 + 8W_{\mathcal{P}}(G)}\right)/2$.

In the next section we present some results which assert that the partition and colored distances of a graph can be obtained from certain smaller weighted graphs. We also show that numerous earlier results.

## 2 Main results

In this section, we investigate a two coloring for hypercube graphs and present a polynomial algorithm for a two coloring of trees.

### Grids

For graphs $G$ and $H$ we let $G \times H$ denote the cartesian product of $G$ and $H$. Recall that the edges $e = xy$ and $f = uv$ of a connected graph $G$ are in the *Djoković-Winkler relation* $\Theta$ [2, 6] if $d_G(x, u) + d_G(y, v) \neq d_G(x, v) + d_G(y, u)$. The transitive closure $\Theta^*$ of $\Theta$ is an equivalence relation on $E(G)$, the corresponding partition is called the $\Theta^*$*-partition* or *cuts*.

Consider a $n$-dimensional hypercube and let $F = \{F_1, \ldots, F_r\}$ be a $\Theta$ partition of $E(G)$. Then each partition $G/F_i$ has $2^{n-1}$ edges. This partition consists of splitters. Then a colouring is ideal iff each $n-1$-dimensional subcube receives equal numbers of red and blue. For example, the bipartite colouring is ideal and hence optimal. But so too is any colouring in which antipodal vertices receive the same colour. (This follows because the requirement for ideal colouring is that in each half of the split there are equal numbers of each colour, and antipodal vertices are always in opposite halves of the split.)

A similar argument holds for any product of even cycles and $K_2$'s with an even number of each colour: any colouring where antipodal vertices receive the same colour is ideal.

Another example is the grid formed by the product of two even paths using equal numbers of red and blue. Then any colouring in which an equal number of reds and blues are placed in each row and in each column is good since it is ideal.

There need not be an ideal colouring, however. For example, consider the grid $P_3 \times P_3$ with $s$-coloring $s = (2, 7)$. If we consider the four natural cuts, then the two reds should always be in the larger component, but this cannot be simultaneously realised.

### Trees

Every tree of even order has an ideal balanced 2-colouring, and the average distance of this 2-colouring is intimately linked to the average distance of the uncoloured tree. Consider the following algorithm:

**Algorithm [1]:** For a tree of even order: Repeatedly choose vertices $u, v$ such that either they are end-vertices at distance 2 or they are adjacent with degree sum at most 3. Colour $u$ and $v$ with different colours and delete.

**Theorem 2.1.** *The above algorithm produces a balanced ideal 2-colouring (with respect to the edge partition where each edge is by itself ).*

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