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Outer independent double Italian domination

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Abstract

An outer independent double Italian dominating function is a double Italian dominating function f for which the set of vertices assigned 0 under f is independent. The outer independent double Italian domination number $\gamma_{oidI}(G)$ is the minimum weight taken over all outer independent double Italian dominating functions of G.

In this work, we characterize Outer independent double Italian domination of some graph cartesian products. Finally, we investigate the families of all graphs G such that $\gamma_{oidI}(G) = |V(G)|$ and for $\delta(G) \geq 2$, the graphs with this property are characterised.

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1 Introduction

In this paper, G is a finite simple graph and $V(G)$ is the vertex set and the edge set is $E(G)$. We use [14] as a reference for terminology and notation which are not explicitly defined here. We denote by $N(v)$ the *open neighborhood* of a vertex v, and $N[v] = N(v) \cup \{v\}$ as *closed neighborhood* of v. We denote by $\delta(G)$ and $\Delta(G)$, respectively for the *minimum* and *maximum degrees* of G. The join of two graphs G and H denoted by $G \vee H$ is a graph obtained from G and H by joining each vertex of G to all vertices of H. For a function $f : V(G) \to \{0, 1, \cdots, k\}$, $V_i = \{v \in V(G) : f(v) = i\}$ and for a subset $S \subseteq V(G)$, $f(S) = \sum_{v \in S} f(v)$.

Definition 1.1. A set $S \subseteq V(G)$ of G is called a *dominating set* if every vertex in $V(G) - S$ has a neighbor in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality among all dominating sets of G.

Definition 1.2. A *Roman dominating function* of a graph G is a function $f : V(G) \to \{0, 1, 2\}$ such that if $f(v) = 0$ for some $v \in V(G)$, then there exists $w \in N(v)$ for which $f(w) = 2$. The *Roman domination number* of G, denoted by $\gamma_R(G)$ is the minimum weight of a Roman dominating function f of G.

This concept was formally defined by Cockayne *et al.* [5] motivated, in some sense, by the article of I. Stewart entitled "Defend the Roman Empire!" ([13]).

Once the original paper [5] was published, many researchers attended to this topics. On the other hand, Beeler *et al.* [4] introduced the concept of double Roman domination.

Definition 1.3. A *double Roman dominating function* (DRD function for short) of a graph G is a function $f : V(G) \to \{0, 1, 2, 3\}$ for which the following conditions are satisfied.

[1]speaker

(a) If $f(v) = 0$, then the vertex v must have at least two neighbors in V_2 or one neighbor in V_3.

(b) If $f(v) = 1$, then the vertex v must have at least one neighbor in $V_2 \cup V_3$.

The double Roman dominating function is called *independent (total) double Roman dominating function* if the induced subgraph $\langle V_1 \cup V_2 \cup V_3\rangle$ has no edges (isolated vertices), [8, 9] and the double Roman dominating function is said to be *an outer independent double Roman dominating function* if the induced subgraph $\langle V_0\rangle$ has no edges or in the other words, the induced subgraph $\langle V_1 \cup V_2 \cup V_3\rangle$ is a covering set [10].

Chellali et al. [**?**] have introduced a Roman $\{2\}$-dominating function (and now is called Italian dominating function) f as follows. A Roman $\{2\}$-dominating function $f : V(G) \to \{0, 1, 2\}$ such that for every vertex $v \in V$, with $f(v) = 0$, $f(N(v)) \geq 2$. In terms of the Roman Empire, this defense strategy requires that every location with no legion has a neighboring location with two legions, or at least two neighboring locations with one legion each. Recently, Mojdeh *et al* [11] introduced the concept of double Italian domination (Roman $\{3\}$-domination).

Definition 1.4. A *double Italian dominating function* (DID function for short) of a graph G is a function $f : V(G) \to \{0, 1, 2, 3\}$ for which the following conditions are satisfied.

(a) If $f(v) = 0$, then the vertex v must have at least three neighbors in V_1 or one neighbor in V_1 and one neighbor in V_2 or one neighbor in V_3.

(b) If $f(v) = 1$, then the vertex v must have at least two neighbors in V_1 or one neighbor in $V_2 \cup V_3$. In the other words, if $v \in V_0 \cup V_1$, then $f(N[v]) \geq 3$.

This parameter was also outlined in [1, 2, 3, 7, 12]. For instance *Total double Italian domination*, which was investigated in [12], *outer independent double Italian domination* which was studied in [1, 2, 3] and *Perfect double Italian domination,* which was studied in [7].

Accordingly, an *outer independent double Italian dominating function* (OIDID function for short) is a DRD function for which V_0^f is independent. The *outer independent double Italian domination number* ($\gamma_{oidI}(G)$) equals the minimum weight of an OIDID function of G.

For the sake of convenience, an OIDID function f of a graph G with weight $\gamma_{oidI}(G)$ is called a $\gamma_{oidI}(G)$-function.

Definition 1.5. The Cartesian product of G and H is the graph $G\Box H$ where $V = V_1 \times V_2$ and two vertices (v_i, u_j) and (v_m, u_n) are adjacent if and only if:
1. $v_i = v_m$ and u_j is adjacent to u_n.
2. $u_j = u_n$ and v_i is adjacent to v_m.

2 OIDID cartesian product in graphs

In this section, we characterize the family of cartesian product of some graphs in terms of outer independent double Italian domination number.

Proposition 2.1. $\gamma_{oidI}(P_m\Box K_{1,n}) = \begin{cases} 2m + \frac{mn}{2}, & \textit{if } n \textit{ is even} \\ 2m + \frac{n+1}{2}\left\lfloor \frac{m}{2}\right\rfloor + \frac{n-1}{2}\left\lceil \frac{m}{2}\right\rceil, & \textit{if } n \textit{ is odd} \end{cases}$

3 Graphs G with OIDID number $|V(G)|$

In this section, we study the graphs with order n in which $\gamma_{oidI}(G) = n$. We start with the following.

Theorem 3.1. *Let G be a graph of order n with $\delta(G) \geq 2$. Then $\gamma_{oidI}(G) = n$ if and only if one of the following conditions holds.*
G is a cycle, or for any vertex v with $\deg(v) = k \geq 3$*, all vertices in $N(v)$ have degree* 2 *and at most one vertex in $N(v)$ like x has a neighbor in $N(x) - N(v)$ like u such that* $\deg(u) \geq 3$

Proof. Let G be a graph with $\delta(G) \geq 2$ and have the given properties. If $\Delta(G) = 2$, then G is a cycle and $\gamma_{oidI}(G) = n$. Let $\Delta(G) \geq 3$ and let $f = (V_0, V_1, V_2, V_3)$ be a γ_{oidI}-function of G. By assignment 1 to all vertices we deduce $\gamma_{oidI}(G) \leq n$. Let v be a vertex and $f(v) = 0$. If $\deg(v) = 2$, then v has a neighbor u with $f(u) \geq 2$. If $\deg(v) \geq 3$, then all vertices in $N(v)$ are of degree 2 and at most one vertex in $N(v)$ like u has a neighbor in $N(u) - N(v)$ like x with $\deg(x) \geq 3$. In this case we should have $f(u) \geq 2$ or $f(x) \geq 2$. Thus for any vertex with value 0 we have a corresponding vertex of value at least 2. Therefore, $\gamma_{oidI}(G) = n$. Conversely, let $\gamma_{oidI}(G) = n$. If a vertex v is of degree $k \geq 3$ and u, w be two vertices in $N(v)$ such that they have neighbors x, y respectively in out of $N(v)$ with $\deg(x) \geq 3$ and $\deg(y) \geq 3$. Now we define $f(v) = 2$, and $f(u) = f(w) = 0$ and $f(z) = 1$ otherwise. Then $w(f) \leq n - 1$. □

Now we investigate a big family of graphs G in which $\gamma_{oidI}(G) = |V(G)|$. It is easy to see that $\gamma_{oidI}(C_n) = n$. For this purpose, we construct the families of graphs in bellow with the given property. Now, we define families of graphs in the form of $\mathcal{G}_i$, $1 \leq i \leq 5$.

- $\mathcal{G}_1$: Let G be a graph such that $G = K \cup H$ and $|V(H)| = 2|V(K)|$, and if $x \in H$, then $|N_H(x)| = 0$, $|N_K(x)| \geq 1$. Also if $y \in K$, then y has at least one private vertex z in H, as well as $|N_K(y)| \in \{0, 1, \ldots, |K| - 1\}$.

- $\mathcal{G}_2$: Let G be a graph such that $G = K \cup H$ and $|V(H)| = |V(K)|$, every vertex in H is adjacent to two vertices in K and if $x \in H$, then $|N_H(x)| = 0$. Also if $y \in K$, then $|N_K(y)| = 0$, $|N_H(y)| \geq 1$, although, maybe there exist some vertex in H such that join to more than two vertices of K.

- $\mathcal{G}_3$: Let G be a block graph where each block be a cycle or an edge between two cycles or $G = C_n$, we can plot at most $\lfloor \frac{n}{4} \rfloor$ chord into C_n, $n \geq 7$ such that no two of the chords share an endpoint and no two of them intersect, and All the cycles that occurred within C_n should be four lengths except the large cycle.

- $\mathcal{G}_4$: Assume that there are three graphs H, K, M such that $G = H \cup K \cup M$, where $|V(H)| = |V(K)| + 2|V(M)|$, and for $x \in H$, $|N_H(x)| = 0$ and if $y \in M$, $|N_M(y)| \geq 0$, $|N_H(y)| \geq 1$, $|N_K(y)| = 0$ and if $z \in K$, $|N_H(z)| \geq 0$, $|N_K(z)| = 0$ such that:

- 4.1. For $x \in H$, if $|N_K(x)| \in \{0, 1\}$, then $|N_M(x)| \geq 1$ and if $|N_K(x)| \geq 2$, then $|N_M(x)| \geq 0$.

- 4.2. Each vertex in M, has a private neighbor in H.

- $\mathcal{G}_5$: Assume that there are three graphs H, K, M such that $G = H \cup K \cup M$, also $|V(H)| = 2|V(M)|$, and for $x \in H$, $|N_H(x)| = 0$, $|N_M(x)| \geq 1$ and if $x \in K$, $|N_K(x)| \geq 1$ and if $x \in M$, $|N_{M \cup K}(x)| \geq 0$, $|N_H(x)| \geq 1$ and each vertex in M, has a private neighbor in H.
  Also let $|V(K)| \neq 0$ and $x \in K$. If $|N_K(x)| = 1$, then for $y \in M$, $x \in N_K(y)$ and for $z \in H$, $x \in N_K(z)$, and if $|N_K(x)| \geq 2$, then $|N_H(x)| \geq 0$ and $|N_M(x)| = 0$.

Theorem 3.2. *Let G be a connected graph. If $G \in \cup_{i=1}^{5} \mathcal{G}_i$, then $\gamma_{oidI}(G) = |V(G)|$.*

Proof. If G in $\mathcal{G}_1$, we assign 3 to the vertices of K and assign 0 to the vertices of H. So $\gamma_{oidI}(G) = n$. If G in $\mathcal{G}_2$, we assign 2 to the vertices of K and assign 0 to the vertices of H which gives $\gamma_{oidI}(G) = n$. If G in $\mathcal{G}_3$ we begin with C_n and connect vertices of it together. In C_5 we can connect just two non adjacent vertices together and in C_6 at most two chords can be drown, provided that no triangle is formed. In $C_n, n \geq 7$ we begin with one chord. If the chord is drown in such a way that the length of the cycle is other than four, we assign 2 to endpoint of the chord and to other vertices assign 0 and 2 alternately, if n be even and assign to one vertex weight 1 and other vertices 0 and 2 alternately, if n is odd. In both cases, we can reassign as one of the endpoint of the chord reduced to 1, therefore $\gamma_{oidI}(G) < n$. So each cycle that is formed must be length four. Now we draw two chords. If two chords have a common vertex we consider a sub graph as follow. Let u be the common vertex and v_1, v_2 are endpoints of two chords and w_1, w_2 adjacent vertices of u and z_1, z_2 adjacent vertices of w_1, w_2 respectively. Then we can assign the graph in which $f(u, v_1, v_2, w_1, w_2, z_1, z_2) = (2, 1, 1, 0, 0, 2, 2)$ and in this case we can reduce weight of v_1 or v_2 to 0 which

gives result $\gamma_{oidI}(G) < n$. If two or more chords intersect, we have two cases about two cycles as follow:
A. Let two cycles are $uvwxu$ and $wxyzw$. We can assign to graph such that $f(u, v, w, x, y, z) = (2, 0, 1, 2, 0, 2)$ and we can reduce weight of the vertex x to one. So in this case $\gamma_{oidI}(G) < n$.
B. Let two cycles are $uvwxu$ and $vwxyw$ and the vertex z is adjacent to y. We can assign to graph such that $f(u, v, w, x, y, z) = (1, 1, 1, 1, 1, 2)$ and we can reduce weight of the vertex y to zero. So in this case $\gamma_{oidI}(G) < n$.
Since the length of the cycles are 4 we do not have more than two cases about chords intersection. Therefore each cycle must be of length four and the cycles should not intersect. On the other hand because each cycles has length 4, so at most $\lfloor \frac{n}{4} \rfloor$ chords can be drawn in C_n. Thus in this case $\gamma_{oidI}(G) = n$. If G in $\mathcal{G}_4$, we assign the value 0 to vertices of the graph H, the value 2 to vertices of the graph K and the value 3 to the vertices of the graph M. In this case, $\gamma_{oidI}(G) = n$. If G in $\mathcal{G}_5$ we assign the value 0 to vertices of graph H, the value 1 to vertices of the graph K and the value 3 to the vertices of the graph M. In this case $\gamma_{oidI}(G) = n$. □

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