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An efficient localized meshless technique for approximating the Sobolev equation

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Abstract

This paper studies a numerical approach for finding the approximate solution of the Sobolev equation. The proposed method approximates the unknown solution with the help of two main stages. At a first stage, the time discretization is performed by means of a second-order finite difference procedure. At a second stage, the space discretization is accomplished using the local radial basis function partition of unity collocation method based on the finite difference (LRBF-PUM-FD). The major disadvantage of global techniques is the high computational burden of solving large linear systems. The LRBF-PUM-FD significantly sparsifies the linear system and reduces the computational burden, while simultaneously maintaining a high accuracy level. Numerical results and comparisons illustrate the high accuracy of the proposed method.

Keywords: Sobolev equation, Finite difference, RBF, LRBF-PUM
Mathematics Subject Classification [2010]: 65M15, 65M60

1 Introduction

In this paper, we focus our attention on the approximated solution of the Sobolev equation [1]:

$$\frac{\partial u(\mathbf{x},t)}{\partial t} - \gamma \frac{\partial \Delta u(\mathbf{x},t)}{\partial t} - \sigma \Delta u(\mathbf{x},t) = f(\mathbf{x},t), \quad \mathbf{x} = (x,y) \in \Omega \subset \mathbb{R}^2, \quad 0 < t \leq T, \tag{1}$$

with boundary conditions

$$u(\mathbf{x},t) = h(\mathbf{x},t), \qquad \mathbf{x} \in \partial\Omega, \quad t > 0 \tag{2}$$

and initial condition

$$u(\mathbf{x},0) = g(\mathbf{x}), \qquad \mathbf{x} \in \overline{\Omega} = \Omega \cup \partial\Omega, \tag{3}$$

where Ω represents a continuous bounded domain in $\mathbb{R}^2$, $\partial\Omega$ denotes the boundary of Ω, T is the final time and Δ stands for the Laplacian operator. The source term $f(\mathbf{x},t)$ is supposed to be sufficiently smooth and two functions $g(\mathbf{x})$ and $h(\mathbf{x},t)$ are given continuous functions. The outline of this paper is as follows. Section 2 implements a second-order of accuracy difference scheme to establish the semi-time discrete formulation. Section 3 presents the LRBF-PUM-FD for the full discretization in the spatial variables. Finally, Section 4 reports the numerical results to show the accuracy and efficiency of the proposed method.

[1] speaker

2 The time-discrete formulation

In this section, we present a Crank-Nicolson formulation to approximate the problem (1) in the temporal direction. For convenience, we require a time step size, which is chosen as $\delta t = T/L$ and $t_k = k\delta t,\ 0 \leq k \leq L$, where L represents the total number of time steps. Let us introduce the notation $u^{k+\frac{1}{2}} = \frac{u^k+u^{k+1}}{2}$, where $\{u^k = u(\mathbf{x}, t_k) | 0 \leq k \leq L\}$ denotes a given grid function. By using the Crank-Nicolson difference scheme, we can get the semi-discrete formulation in the time variable

$$\frac{u^{k+1}-u^k}{\delta t} - \gamma \frac{\Delta u^{k+1} - \Delta u^k}{\delta t} - \sigma \frac{\Delta u^{k+1} + \Delta u^k}{2} = f^{k+\frac{1}{2}} + \mathcal{R}^{k+\frac{1}{2}}, \tag{4}$$

or

$$u^{k+1} - \left(\gamma + \sigma \frac{\delta t}{2}\right) \Delta u^{k+1} = u^k - \left(\gamma - \sigma \frac{\delta t}{2}\right) \Delta u^k + \delta t f^{k+\frac{1}{2}} + \delta t \mathcal{R}^{k+\frac{1}{2}}, \tag{5}$$

where $u^k = u(\mathbf{x}, t_k)$ represents the analytic solution and $\mathcal{R}^{k+\frac{1}{2}}$ is the local truncation error, which is bounded by

$$\left|\mathcal{R}^{k+\frac{1}{2}}\right| \leq C\delta t^2, \quad \text{or} \quad \mathcal{R}^{k+\frac{1}{2}} = \mathcal{O}(\delta t^2),$$

where C is positive constant. Dropping the truncation error $\mathcal{R}^{k+\frac{1}{2}}$ from the above relation and introducing $U^k = U(\mathbf{x}, t_k)$ as the approximation solution of u^k, we derive the following semi-discrete scheme

$$U^{k+1} - \left(\gamma + \sigma \frac{\delta t}{2}\right) \Delta U^{k+1} = U^k - \left(\gamma - \sigma \frac{\delta t}{2}\right) \Delta U^k + \delta t f^{k+\frac{1}{2}}. \tag{6}$$

3 The spatial discretization

The RBF interpolation method uses linear combination of translates of one function $\phi(r)$ of a single real variable. Given a set of centers Ξ in Ω, the RBF interpolant takes the form

$$u(\mathbf{x}) \simeq S(\mathbf{x}) = \sum_{j=1}^{N} \lambda_j \phi_j(\mathbf{x}, \varepsilon), \tag{7}$$

where λ_j are unknown real coefficients, $\phi_j(\mathbf{x}, \varepsilon) = \phi(\|\mathbf{x} - \mathbf{x}_j\|_2, \varepsilon)$, $j = 1, \ldots, N$, is a smooth strictly positive definite (SPD) RBF depending on a shape parameter ε, and $r = \|\mathbf{x} - \mathbf{x}_j\|_2$ denotes the Euclidean norm. Imposing the interpolation conditions $S(\mathbf{x}_i) = u_i$, $i = 1, \ldots, N$ results in the linear system

$$A\lambda = u, \tag{8}$$

in which $A_{ij} = \phi(\|\mathbf{x}_i - \mathbf{x}_j\|_2, \varepsilon)$, $\lambda = [\lambda_1, \ldots, \lambda_N]^T$, and $u = [u(\mathbf{x}_1), \ldots, u(\mathbf{x}_N)]^T$. The alternative formulation for the interpolant (7) is derived as follows:

$$S(\mathbf{x}) = \Psi(\mathbf{x}) u, \tag{9}$$

in which $\Psi(\mathbf{x}) = [\psi_1(\mathbf{x}), \ldots, \psi_N(\mathbf{x})]^T$. By using Eqs. (9), (8) and (7), we can deduce the relation between primary radial basis and cardinal basis as:

$$\Psi(\mathbf{x}) = \Phi A^{-1}, \tag{10}$$

in which $\Phi = [\phi(\|\mathbf{x} - \mathbf{x}_1\|_2, \varepsilon), \ldots, \phi(\|\mathbf{x} - \mathbf{x}_N\|_2, \varepsilon)]^T$ [2]. Let $\Omega \subset \mathbb{R}^2$ be a bounded set, and let a covering $\{\Omega_j\}_{j=1}^{M}$ of the region Ω such that $\bigcup_{j=1}^{M} \supseteq \Omega$. Moreover, sub-domains Ω_j satisfy some mild overlap condition. This means that each point of the global computational domain must be in the interior of at least one local subdomain. Further, the set $I(\mathbf{x}) = \{j : \mathbf{x} \in \Omega_j\}$, for all $\mathbf{x} \in \Omega$, is uniformly bounded on Ω, i.e., there is the constant C independent of the number of sub-domains, such that $\mathrm{card}(I)(\mathbf{x}) \leq C$. We can establish PU weight function w_j for each sub-domain Ω_j with the help of the Shepard method as

$$w_j(\mathbf{x}) = \frac{\varphi_j(\mathbf{x})}{\sum\limits_{k\in I(\mathbf{x})} \varphi_k(\mathbf{x})}, \tag{11}$$

where $\varphi_j(\mathbf{x})$ is the compactly supported function on Ω_j. The weight functions w_j are non-negative, compactly supported on Ω_j, and satisfy the PU property, i.e.,

$$\sum w_j(\mathbf{x}) = \begin{cases} 0, & j \notin I(\mathbf{x}), \\ 1, & j \in I(\mathbf{x}). \end{cases} \tag{12}$$

In addition, to ensure that weight function is non-negative and compactly support on Ω_j, the function $\varphi_j(\mathbf{x})$ is defined as follows:

$$\varphi_j(\mathbf{x}) = \varphi_j(\frac{\|\mathbf{x} - \tilde{\mathbf{x}}_j\|}{R_j}), \qquad j = 1, 2, \ldots, M, \tag{13}$$

where $\tilde{\mathbf{x}}_j$ and R_j are the center and radius corresponding to the j-th sub-domain, respectively, and φ_j denotes the Wendland C^2 function for constructing the weight function. The PUM approximation is formed a global approximation function $\mathcal{P}$ of function $u(\mathbf{x})$ in entire domain Ω as follows:

$$\mathcal{P}_u(\mathbf{x}) = \sum_{j\in I(\mathbf{x})} w_j(\mathbf{x}) S_j(\mathbf{x}) = \sum_{j\in I(\mathbf{x})} \sum_{i\in J(\Omega_j)} w_j(\mathbf{x}) \psi_i(\mathbf{x}) u(\mathbf{x}), \tag{14}$$

where $\{S_j\}_{j=1}^{N}$ are RBF based local interpolants corresponding to each sub-domain Ω_j. From (14), the approximation solution U^k can be expressed as

$$U^k \approx \mathcal{P}_u(\mathbf{x}) = \sum_{j\in I(\mathbf{x})} \sum_{i\in J(\Omega_j)} w_j(\mathbf{x}) \psi_i(\mathbf{x}) U^k. \tag{15}$$

4 Numerical results

Figure 1 includes a representation of the associated PU subdomains for the considered domains with the uniform and Halton points. In the present research, the subdomains are selected as circles in $\mathbb{R}^2$, but they can selected also in other forms, such as ellipses or squares. The box structure presented by [3] is used here to define the centers and radius of the subdomains. In this paper, we follow the algorithm provided by Sarra to specify an optimal shape parameter ε. The value of condition number (CN) is calculated with the help of the Matlab command `condest`. The numerical experiments were done in `MATLAB 2016a` on a PC with 8 GB of RAM.

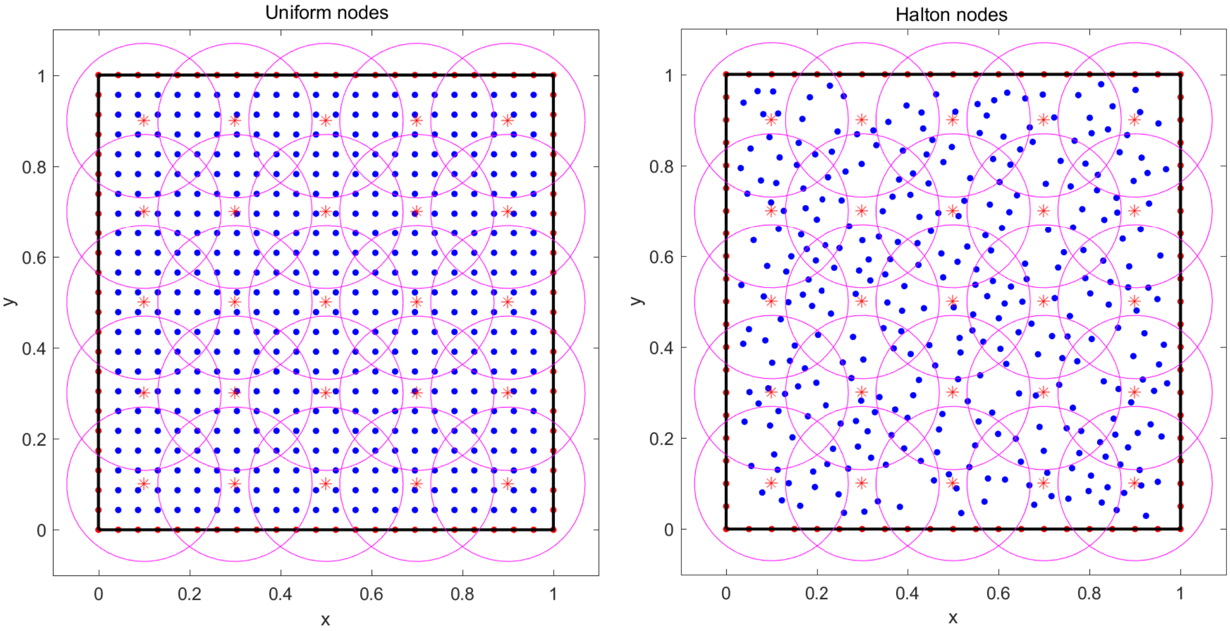

Figure 1: Partitioning of considered domains having circular subdomains.

Consider the following Sobolev equation

$$\frac{\partial u(x,y,t)}{\partial t} - \gamma \frac{\partial \Delta u(x,y,t)}{\partial t} - \sigma \Delta u(x,y,t) = f(x,y,t), \qquad (x,y) \in \Omega, \qquad 0 < t \leq T,$$

The source term $f(x,y,t)$, the initial and Dirichlet boundary conditions are calculated from the analytical solution $u(x,y,t) = \exp(-t)\sin(\pi x)\sin(\pi y)$. Table 1 lists the L_2 errors, the time convergence orders $C_{\delta t}$ and the obtained computational times (in seconds) of the proposed method by taking $N = 100$ at final time $T = 1$ on the square domain $\Omega = [0,1]^2$. Table 2 makes the comparison of the L_∞ and L_2 errors with those provided by the techniques described in [5–7] on the square domain $\Omega = [0,1]^2$. The obtained numerical results clarify that the proposed method is better than the others in terms of the accuracy and the elapsed computational time. Table 3 displays the L_∞ errors, the condition number and the associated computational times on the square domain $\Omega = [0,1]^2$ for the uniform and Halton nodes with $\delta t = 1/100$.

Table 1: Numerical errors, time convergence orders $C_{\delta t}$ and associated computational time of LRBF-PUM-FD with $\gamma = \sigma = 1$ and $N = 100$ at $T = 1$ on the domain $\Omega = [0,1]^2$.

δt	Ref. [4]		LRBF-PUM-FD		
	L_2	CPU	L_2	$C_{\delta t}$	CPU
1/2	$1.1963e-01$	0.89	$7.8104e-03$	–	0.424952
1/4	$1.4943e-02$	1.77	$2.8091e-03$	1.47529	0.443898
1/8	$7.2957e-03$	3.44	$6.9891e-04$	2.00693	0.444320
1/16	$1.8217e-03$	6.78	$1.7452e-04$	2.00171	0.451735
1/32	$4.5530e-04$	13.45	$4.3616e-05$	2.00047	0.464944
1/64	$1.1382e-05$	26.91	$1.0903e-05$	2.00013	0.475786
1/128	$2.8453e-05$	54.44	$2.7258e-06$	1.99998	0.482230

Table 2: Comparison between the L_∞ and L_2 errors of the LRBF-PUM-FD and the methods of [5–7] with $\delta t = 1/100$ and $\gamma = \sigma = 1$ on the domain $\Omega = [0,1]^2$.

	Ref. [5]		Ref. [6]		Ref. [7]		LRBF-PUM-FD	
N	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2
4	$2.2100e-02$	$8.0968e-03$	$6.4901e-03$	$9.3030e-04$	$7.4767e-03$	$1.7519e-02$	$1.1566e-06$	$2.3132e-06$
8	$6.4901e-03$	$2.9283e-03$	$1.4656e-04$	$4.9647e-04$	$2.1515e-03$	$8.9465e-03$	$2.9784e-06$	$3.8763e-06$
16	$1.7223e-03$	$8.0121e-04$	$1.2647e-05$	$9.8424e-05$	$4.9543e-04$	$4.0019e-03$	$3.1065e-06$	$5.8890e-06$
32	$4.2893e-04$	$1.9983e-04$	$3.7398e-06$	$5.9866e-05$	$5.9549e-05$	$9.5508e-04$	$3.1328e-06$	$8.6059e-06$

Table 3: The L_∞ errors, condition numbers and associated computational times on the square domain with $\gamma = \sigma = 1$ and $\delta t = 1/100$ at $T = 1$ on the domain $\Omega = [0,1]^2$.

Uniform nodes					Halton nodes				
N	M	L_∞	C-number	CPU	N	M	L_∞	C-number	CPU
289	9	$3.1400e-06$	$7.2666e+04$	1.45537	316	10	$3.1232e-06$	$9.7179e+04$	5.78059
676	16	$3.1285e-06$	$2.7648e+04$	3.86449	639	18	$3.1282e-06$	$5.1763e+05$	3.49469
1024	36	$3.1416e-06$	$5.5667e+05$	5.84741	1729	29	$3.1519e-06$	$1.1767e+06$	6.03137

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