Perfect 2-colorings of the quartic graphs of order 9

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Abstract

In this paper, we study perfect 2-coloring of the quartic graphs with 9 vertices. The problem of the existence of perfect coloring is a generalization of the concept of completely regular codes, given by Delsarte.

Keywords: Perfect coloring, Equitable partition, Quartic graph, Perfect code, Regular graph

Mathematics Subject Classification [2010]: 05C15 , 05E30

1 Introduction

Let $X = (V(X), E(X))$ be a connected graph. A vertex coloring with m colors is perfect if there is a matrix $M = (\alpha_{ij})_{i,j=1,2,\cdots,m}$ such that the cardinal of vertices of label j connected to a specified vertex of label i is equal to $\alpha_{i,j}$.

In particular, all vertices of the same color in X have the same degree. This type of coloring gives a partition of $V(X)$ that in called **equitable partitions**. These partitions of vertices have two properties as follows: The first property is that the vertices of each part P_i induce a regular graph, and the second is the edges between P_i and P_j induce a half-regular graph. In Section 3, we find the parameter matrices of a perfect 2-coloring. Equitable partitions were previously studies in [7].

In fact, the concept of a perfect $m-$coloring is bridge of among algebraic combinatorics, graph theory and coding theory (including 1-perfect codes). For the definition of 1-perfect codes and more details, we refer the reader to [1].

This type of coloring is generalized the concept of completely regular codes introduced by P. Delsarte ([8]). The problem for the Johnson graphs has been studied (see [4], see [3], see [6]).

In subsequent attempts, the problem of existence of perfect coloring was examined on other graphs. For example, Fon-Der-Flass settled perfect coloring with two colors of $n-$dimensional hypercube graphs Q_n for $n < 24$ (see [5]).
In this paper, we will list all parameter matrices of perfect 2-coloring of the quartic graphs of order 9.

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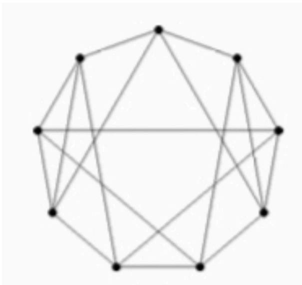
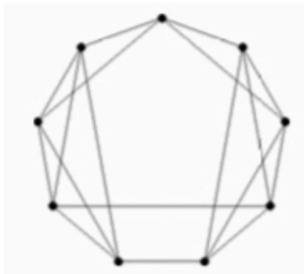
2 Notations and preliminaries result on the existence of perfect coloring

In this section, some basic definitions that will be used in this paper are given and we declare some conditions for existence of perfect 2-coloring of the quartic graphs.

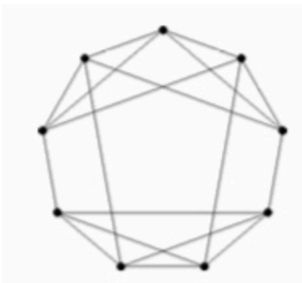
In this paper, all the graphs are simple. A graph $X = (V(X), E(X))$ is k-regular if for every $v \in V(X)$ we have $\deg(v) = k$, where k is a nonnegative integer number. In particular, graphs that are regular of degrees 3,4,5 and 6 are called, respectively, cubic, quartic, quantic and sixtic. The quartic graphs of order 9 are shown in Figure 1.



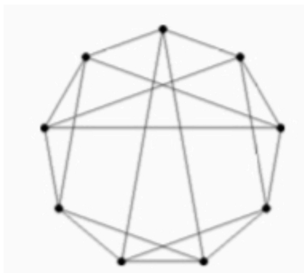
(a) K_2



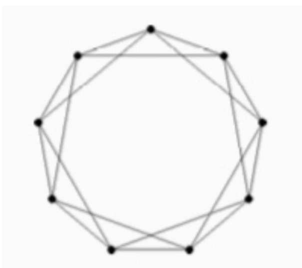
(b) K_1



(c) K_4



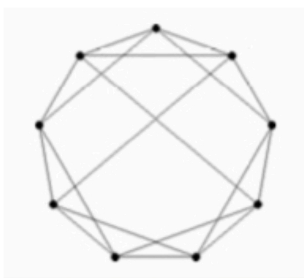
(d) K_3



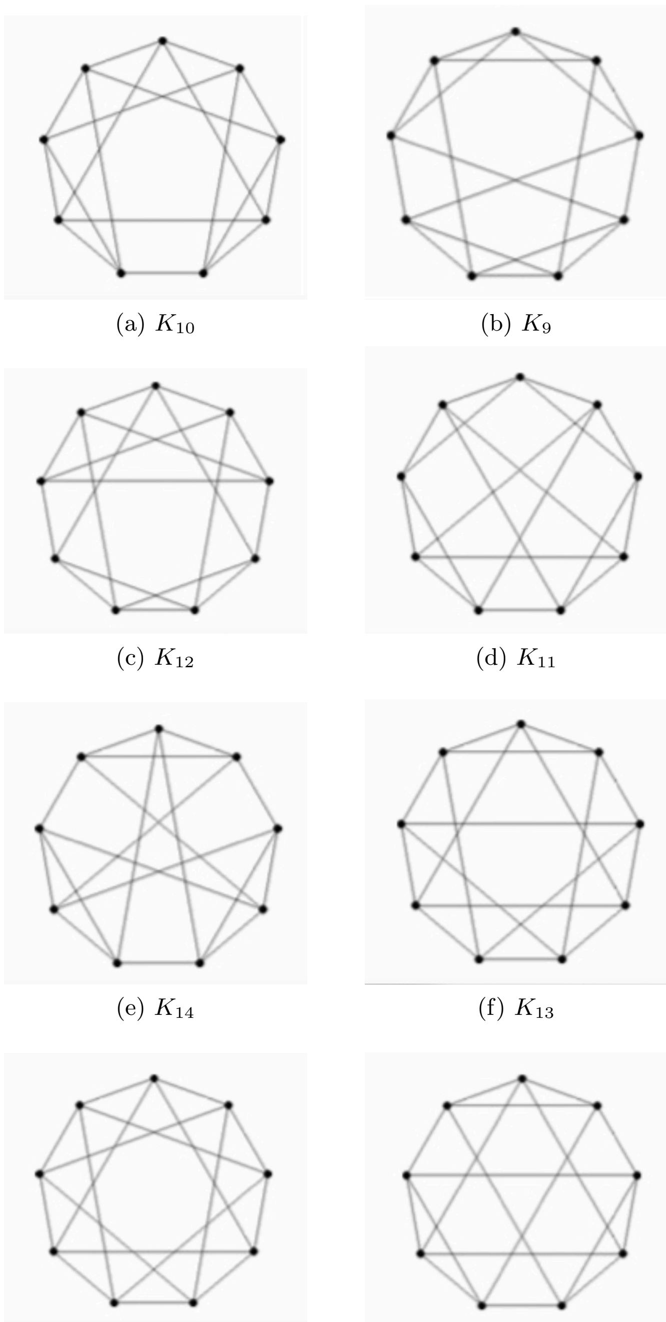
(e) K_6

(f) K_5

(g) K_8



(h) K_7



(a) K_{10} (b) K_9

(c) K_{12} (d) K_{11}

(e) K_{14} (f) K_{13}

(g) K_{16} (h) K_{15}

Figure 1: Connected quartic graphs of order 9

Definition 2.1. Let $X = (V(X), E(X))$ be a connected graph. A perfect $m-$coloring with parameter matrix $M = (\alpha_{ij})_{i,j=1,2,\cdots m}$ is a map C from $V(X)$ to the set of colors $\{1, 2, \cdots, m\}$ such that C is surjective, and for a fix vertices v where $C(v) = i$ we have $|\{w \in V(x)|w$ is adjacent to v, $C(w) = j\}| = \alpha_{ij}$, for all i, j. In this case, we say C is a P.m.C (or just P.C). Here, we study P.2.C, where two colors are red and blue. In this case, we consider the parameter matrix of P.2.C generated by replacing the colors with the primary coloring is equal. This means the parameter matrix $\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$ is equal to the parameter matrix $\begin{pmatrix} b' & b \\ a' & a \end{pmatrix}$. We call **equality of parameter matrix**. Some properties of P.C have been studied recently (see [4, 6, 8]). The main result of them is the next proposition that enumerates the cardinal number of blue vertices in a P.2.C with matrix $M = (\alpha_{ij})_{i,j=1,2}$ (see [3]).

Proposition 2.2. *Assume that B is the set of all blue vertices in a P.2.C of a graph $X = (V(X), E(X))$*

with matrix $\begin{pmatrix} a & b \\ a^{'} & b^{'} \end{pmatrix}$.
Then we have

$$|B| = \frac{a^{'}|V(X)|}{b+a^{'}}. \tag{1}$$

Now, the following remark provides useful information about calculation P.2.C of the quartic graphs.

Remark 2.3. Suppose that $X = (V(X), E(X))$ be a simple connected graph. Then necessary conditions for existence of P.2.C of with parameter matrix $M = \begin{pmatrix} a & b \\ a^{'} & b^{'} \end{pmatrix}$ are listed as follows:

1. Degree condition: This simple condition that results from the regularity of tells us:

$$a + b = a^{'} + b^{'} = 4.$$

2. Connectedness condition: Since the graph X is connected, another condition is induced as follows:

$$b, a^{'} \neq 0.$$

We will define a new definition for the third condition. The number μ is called an **eigenvalue of a graph** X, if μ is an eigenvalue of the adjacency matrix of X. The number μ is called an **eigenvalue of a** P.2.C with the martix M, if μ is an eigenvalue of M.

The next theorem gives us the third condition and the connection between two types of eigenvalues (see [4]):

Theorem 2.4. *Let C be a P.m.C of a graph X. Thus any eigenvalues of C are an eigenvalues of X.*

It has been proved that a P.2.C of a $k-$regular graph X has just two eigenvalues (see [4]).

Proposition 2.5. *If C be a P.2.C with parameters matrix $M = (\alpha_{i,j})_{i,j=1,2}$ of a regular graph X of valency k, then the numbers $\alpha_{11} - \alpha_{21}$ and k are eigenvalues of C and therefore are eigenvalues of X.*

Lemma 2.6. *Given an arbitrarily connected quartic graph X, the following ten matrices are the only matrices that can be selected as* ***acceptable*** *parameter matrices of a perfect 2-coloring C of X:*

$$M_1 = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 4 \\ 3 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 4 \\ 2 & 2 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 0 & 4 \\ 1 & 3 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix},$$

$$M_6 = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}, \quad A_7 = \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

$$A_9 = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}, \quad M_{10} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Proof. By Remark (2.3), Theorem (2.4), Proposition (2.5) and equality of parameter matrix, proof is clear. □

Note that the word **acceptable** in the previous lemma states that some of the listed matrices may not be a parameter matrix of a P.2.C of X. Therefor, we reduce them by Proposition (2.2) in the next lemma (Table 1):

Table 1: Acceptable parameter matrices of quartic graphs with order 9

Matrices	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}
$\lvert B\rvert$	4/5	3/8	3	1/8	4/5	3/6	2/2	4/5	3	4/5
Acceptability	×	×	√	×	×	×	×	×	√	×

Lemma 2.7. *Let X be the quartic graph with 9 vertices. According to order of X, we can reduce the listed parameter matrices in Lemma* (2.6) *as follows:*

Proof. Since the number of blue vertices ($|B|$) is not an integer, there is not exit a P.2.C of X with the given parameter matrix M_i. □

Corollary 2.8. *If X be a connected quartic graph with 9 vertices, then only the matrices listed in the following table are acceptable as parameter matrices of a P.2.C of X (Table 2):*

Table 2: Acceptable parameter matrices of quartic graph with order 9

Graphs	Order 9
Matrices	M_3, M_9

3 Perfect 2-coloring of the quartic graphs of order 9

In the previous section, we determined all acceptable parameter matrices of a P.2.C of a connected quartic graph with 9 vertices, In this section, we show which of them have a **structure**. In the other words, we introduce a perfect 2-coloring C for them.

Theorem 3.1. *The parameter matrices of a P.2.C of a quartic graph of order 9 are listed in the following table:*

Proof. From Corollary (2.8), we conclude that acceptable parameter matrices of the quartic graphs with order 9 are M_3 and M_9. First, we determine structures for the quartic graphs with order 9 with their listed parameter matrices in Table 3 as follows:

Table 3: Parameter matrices of the quartic graphs of order 9

Matrices graphs	M_3	M_9
K_1	$\surd$	$\times$
K_2	$\times$	$\times$
K_3	$\times$	$\times$
K_4	$\times$	$\times$
K_5	$\surd$	$\times$
K_6	$\surd$	$\times$
K_7	$\surd$	$\times$
K_8	$\times$	$\times$
K_9	$\times$	$\times$
K_{10}	$\times$	$\surd$
K_{11}	$\surd$	$\surd$
K_{12}	$\times$	$\times$
K_{13}	$\times$	$\surd$
K_{14}	$\surd$	$\surd$
K_{15}	$\times$	$\surd$
K_{16}	$\times$	$\surd$

Structure for K_1 with matrix M_3:

$$C(v_1) = C(v_4) = C(v_6) = 1, \quad C(v_2) = C(v_3) = C(v_5) = C(v_7) = C(v_8) = C(v_9) = 1,$$

Structure for K_5 with matrix M_3:

$$C(v_1) = C(v_4) = C(v_6) = 1, \quad C(v_2) = C(v_3) = C(v_5) = C(v_7) = C(v_8) = C(v_9) = 2,$$

Structure for K_6 with matrix M_3:

$$C(v_1) = C(v_4) = C(v_7) = 1, \quad C(v_2) = C(v_3) = C(v_5) = C(v_6) = C(v_8) = C(v_9) = 2,$$

Structure for K_7 with matrix M_3:

$$C(v_1) = C(v_4) = C(v_7) = 1, \quad C(v_2) = C(v_3) = C(v_5) = C(v_6) = C(v_8) = C(v_9) = 2,$$

Structure for M_{10} with matrix M_9:

$$C(v_1) = C(v_4) = C(v_7) = 1, \quad C(v_2) = C(v_3) = C(v_5) = C(v_6) = C(v_8) = C(v_9) = 2,$$

Structure for M_{11} with matrix M_3:

$$C(v_1) = C(v_5) = C(v_7) = 1, \quad C(v_2) = C(v_3) = C(v_4) = C(v_6) = C(v_8) = C(v_9) = 2,$$

Structure for M_{11} with matrix M_9:

$$C(v_1) = C(v_2) = C(v_3) = 1, \quad C(v_j) = 2, \quad for\ j = 4, 5, \cdots, 9,$$

Structure for M_{13} with matrix M_9:

$$C(v_1) = C(v_2) = C(v_9) = 1, \quad C(v_3) = C(v_4) = C(v_5) = C(v_6) = C(v_7) = C(v_8) = 2,$$

Structure for M_{14} with matrix M_3:

$$C(v_1) = C(v_4) = C(v_7) = 1, \quad C(v_2) = C(v_3) = C(v_5) = C(v_6) = C(v_8) = C(v_9) = 2,$$

Structure for K_{14} with matrix M_9:

$$C(v_1) = C(v_2) = C(v_9) = 1, \quad C(v_3) = C(v_4) = C(v_5) = C(v_6) = C(v_7) = C(v_8) = 2,$$

Structure for K_{15} with matrix M_9:

$$C(v_1) = C(v_2) = C(v_9) = 1, \quad C(v_3) = C(v_4) = C(v_5) = C(v_6) = C(v_7) = C(v_8) = 2,$$

Structure for K_{16} with matrix M_9:

$$C(v_3) = C(v_6) = C(v_9) = 1, \quad C(v_1) = C(v_2) = C(v_4) = C(v_5) = C(v_7) = C(v_8) = 2.$$

It is easily to see that the above functions are perfect 2-coloring with their expressed parameter matrices. Now, we show there are no P.2.C of $K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8, K_9$ and K_{12} with matrix M_9. On the contrary, suppose that C is a perfect 2-coloring of K_2 with the matrix M_9. Let $C(v_1) = 1$ and $C(v_2) = 2$. From $a_{21} = 1$, we have $C(v_3) = C(v_4) = C(v_5) = 2$. On the other hand, from $a_{11} = 2$, we obtain $C(v_7) = C(v_9) = 1$. We now have there possibilities:
(1) $C(v_6) = 1\ and\ C(v_8) = 2$,
(2) $C(v_6) = 2\ and\ C(v_8) = 1$.
(2) $C(v_6) = C(v_8) = 2$.
In cases 1 and 2 we get a contradiction with $|B| = 3$. In other case we get a contradiction with the second row of M_9. Hence K_2 has no perfect 2-coloring with matrix M_9. The proof for $K_1, K_3, K_4, K_5, K_6, K_7, K_8, K_9$ and K_{12} with matrix M_9 is Similar.
Now, we show there are no P.2.C of $K_2, K_3, K_6, K_8, K_9, K_{10}, K_{12}, K_{13}, K_{15}$ and K_{16} with matrix M_3. On the contrary, suppose that C is a perfect 2-coloring of K_3 with the matrix M_3.
Let $C(v_1) = 1$. From a_{21}, we have $C(v_2) = C(v_5) = C(v_6) = C(v_9) = 2$. We now have two possibilities:
(1) $C(v_3)\ and\ C(v_4) = C(v_8) = 2$,
(2) $C(v_3) = 2\ and\ C(v_4) = C(v_8) = 1$.
In case 1 we get a contradiction with the second eow of M_3. In other case we get $C(v_7) = 2$, which is a contradiction with the second row of matrix M_3. About other graphs in Figure 1, Similary, we can get the same result as in Table 3. □

Corollary 3.2. *The only P.2.C of the quartic graphs with 9 vertices is prefect coloring with listed matrices in the following table: (Table 4).*

Table 4: All parameter matrices of the quartic graphs of order 9

Graphs	parameter matrices of a perfect 2-coloring
K_1	M_3
K_2	The K_2 has no any perfect 2-coloring
K_3	The K_3 has no any perfect 2-coloring
K_4	The K_4 has no any perfect 2-coloring
K_5	M_3
K_6	M_3
K_7	M_3
K_8	The K_8 has no any perfect 2-coloring
K_9	The K_9 has no any perfect 2-coloring
K_{10}	M_9
K_{11}	M_3, M_9
K_{12}	The K_{12} has no any perfect 2-coloring
K_{13}	M_9
K_{14}	M_3, M_9
K_{15}	M_9
K_{16}	M_9

4 Conclusion

The perfect coloring of graphs is closely related to coding theory, algebraic theory, graph theory and combinatorics, including designs, We can consider perfect $m-$coloring as a generalization of the concept of completely regular codes presented by P. Delsarte for the first time. This class of codes has been of interest to coding theorists and graph theorists alike. In this way, the problem of existence of perfect coloring of the Johnson graphs, the generalized Petersen graphs and the $n-$dimensional hypercube graphs Q_n for $n < 24$ has been settled in [[2]-[4], [6]] and [5].

In this paper, we have listed all parameters of existing perfect 2-coloring in the quartic graphs with order 9. The question of existence of perfect 3-coloring of the quartic graphs remains open.

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